$\#(\bigcup_{s\in\omega} A(K'_\alpha, \alpha, s))<c$ and $\exists p_\alpha \in C_\alpha \setminus \bigcup A(K'_\alpha, \alpha, s)$. Let $K_\alpha = K'_\alpha \cup \{p_\alpha\}$. Then K_α fulfills the inductive assumptions and $K=\bigcup K_\alpha$ satisfies dim $K^s = n-1$, for each $s\in\omega$. Hence, by Lemma 4, dim $K^\omega < n$. But $K^\omega \supset K'$ where $K' \overset{T}{\approx} K$ and dim $K \geqq n-1$. Thus dim $K^\omega = n-1$.

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IMMERSIONS INTO MANIFOLDS OF CONSTANT NEGATIVE CURVATURE

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1. **Introduction.** Let M and $\overline{M}$ denote C^∞ Riemannian manifolds, K and $\overline{K}$ their respective sectional curvature functions, and $\psi: M \to \overline{M}$ an isometric immersion. A consequence of Theorem 2 of [5] is that if at any point $m \in M$, $K(\pi) < \overline{K}(d\psi(\pi))$, where π is some plane in M_m, (the tangent space to M at m) then there are no ψ that immerse M^d in $\overline{M}^{d+k}$ unless k is greater than or equal to $d-1$. By restricting M to be compact and $\overline{M}$ to be complete and simply connected, O'Neill has shown in [3] that there are no isometric immersions of M^d in $\overline{M}^{d+k}$ when $K \leqq \overline{K} \leqq 0$ on M unless k is greater than or equal to d. Amaral (Theorem A of [1]) considered immersions of compact M^d in $H^{d+1}(\overline{C})$, $(d+1)$-dimensional hyperbolic space of curvature $\overline{C}$, and by only assuming $K \leqq 0$ proved that there are no isometric immersions of M^d in $H^{d+1}(\overline{C})$. Using methods similar to those of [3] we prove a theorem which strengthens O'Neill's result in the case that $\overline{M}$ is of constant negative curvature and includes Amaral's result.

2. **Results.**

THEOREM. *Let M be a compact d-dimensional Riemannian manifold and let $\overline{M}$ be a complete simply connected Riemannian manifold of constant curvature $\overline{C} \leqq 0$ and of dimension less than $2d$. If the sectional*

Received by the editors July 13, 1966.

[1] Part of the research in this paper was done while the author was the recipient of an NSF Research Participation award in Mathematics at the University of Oklahoma.

curvature function of M satisfies $K \leqq 0$, then M cannot be isometrically immersed in $\overline{M}$.

There exist immersions of $H^d(C)$ in $H^{2d-1}(\overline{C})$ for any $\overline{C} < C \leqq 0$ (where $H^d(0) = R^d$, d-dimensional Euclidean space); our theorem shows there are no compact ones. It is also known that the flat torus T^d can be isometrically immersed in $H^{2d}(\overline{C})$ for any $\overline{C} < 0$; the theorem above shows a similar immersion of T^d cannot be found in $H^{2d-1}(\overline{C})$.

We express the information contained in the second fundamental form operators of the immersion ψ by means of the operators $T_x \colon \overline{M}_m \to \overline{M}_m$ where $x \in M_m$ (see page 191 of [2]). The T_x are skew-symmetric, bilinear, and interchange $d\psi(M_m)$ and the orthogonal complement of $d\psi(M_m)$ in $\overline{M}_m$, $M_m^{\perp}$. To simplify notation, we identify M_m and $d\psi(M_m)$ when it seems safe to do so. In terms of these operators the Gauss equation for an immersion becomes $K(\pi) = \Delta_{xy} + \overline{K}(d\psi(\pi))$ where x and y span π and

$$\Delta_{xy} = (\langle T_x x, T_y y \rangle - |T_y x|^2)/|x \wedge y|^2.$$

The real valued function defined by $f(x) = |T_x x|$ for unit $x \in M_m$ has the following properties (see Lemmas 2 and 3 of [4]): *If y is a critical point of f such that $f(y) \neq 0$, then $\langle T_y y, T_y x \rangle = 0$ for all x in M_m that are perpendicular to y. Also $T_y x = 0$ implies $\langle x, y \rangle = 0$. If y is a minimum of f such that $f(y) \neq 0$ and x is a unit vector orthogonal to y then $\Delta_{xy} + 3|T_y x|^2 \geqq |T_y y|^2$.* We shall also use the classical result that the curvature operators of the constant curvature manifold $\overline{M}(\overline{C})$ have the form $R_{xy} z = \overline{C}\{\langle x, z \rangle y - \langle y, z \rangle x\}$.

PROOF OF THEOREM. Because of the result quoted above from [3], we need only consider the case in which $\overline{C} < 0$. We assume an immersion $\psi \colon M^d \to \overline{M}^{d+k}(\overline{C})$ exists with $k < d$ and arrive at a contradiction. Let $\overline{m}$ be a fixed point in $\overline{M}$ and let $\psi(m) \in M$ be such that the distance from $\overline{m}$ to $\psi(m)$ is a maximum. Let $\sigma[0, b] \to \overline{M}$ be the unique unit speed geodesic from $\overline{m}$ to $\psi(m)$ and z the velocity vector, $\sigma'(b)$, of σ at $\psi(m)$. For any unit x in $d\psi(M_m)$ there exists a differentiable $r \colon [0, b] \times [0, 1] \to \overline{M}$ such that: (1) $r(\cdot, 0) = \sigma$; (2) for each $v \in [0, 1]$, $r(\cdot, v)$ is a geodesic; (3) $r(0, \cdot) = \overline{m}$ and $r(b, \cdot)$ is in $\psi(M)$; (4) if X is the vector field on σ such that $X(u)$ is the velocity of $r(u, \cdot)$ at $v = 0$, then $X(0) = 0$ and $X(b) = x$. Let $l(v)$ be the length of $r(\cdot, v)$; i.e. the distance from $\overline{m}$ to $r(b, v) \in \psi(M)$. Since $l(0) = 0$ it follows that $z \in M_m^{\perp}$ and since 0 is a maximum of l, $l''(0) \leqq 0$. From the Synge formula for the second variation (cf. page 219 of [2]), we get

$$l''(0) = \int_0^b \{ |X'(u)|^2 - \overline{C} |X(u)|^2 \} du + \langle T_x x, z \rangle \leqq 0.$$

The $r(\cdot, v)$ curves are geodesics of $\overline{M}$ and therefore X is a Jacobi field and satisfies the Jacobi equation $X'' = R_{X\sigma'}(\sigma') = \overline{C}(\langle X, \sigma'\rangle\sigma' - \langle\sigma', \sigma'\rangle X) = -\overline{C}X$, the last equality following since $\langle X, \sigma'\rangle$ is constant along σ and 0 at b. This implies that $X(u) = (\sinh cu)P(u)$, where $c = \sqrt{(-\overline{C})}$, P is a parallel field on σ, and $X(b) = (\sinh cb)P(b) = x$. The last relation implies that $|P(u)| = (\sinh cb)^{-1}$. Substituting these values for $X(u)$ and $|P(u)|$ into the Synge formula and integrating we find that $\langle T_x x, z\rangle \leqq -(c/2)(\sinh 2bc)(\sinh^2 bc)^{-1} = -c \coth bc < -c$.

The function $f(x) = |T_x x|$ for unit $x \in M_m$ takes on a minimum at, say, y. This minimum, since $\langle T_y y, z\rangle < -c$, exceeds c. Also, since $k < d$, there exists a unit x in M_m such that $T_y x = 0$. By the results quoted from [4] we must have that $\langle x, y\rangle = 0$ and also that $\Delta_{xy} + 3|T_y x|^2 \geqq |T_y y|^2$ or $\Delta_{xy} > c^2 = |\overline{C}|$. However, $K = \Delta_{xy} + \overline{C} > |\overline{C}| + \overline{C} = 0$. This contradicts the hypothesis that $K \leqq 0$ and hence the proof of the theorem is complete.

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