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AN EXAMPLE OF TWO UNIFORMITIES EQUAL IN HEIGHT AND PROXIMITY

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1. Introduction. In [7] Yu. M. Smirnov first raised the question whether there existed a proximity class without a largest member. The first example was given in [2] and later examples occurred in [4] and [6].

In [5] the author introduced the concept of height of uniformities and showed that if two uniformities in the same proximity class were comparable in height but not comparable in the usual ordering, then their least upper bound was not in that proximity class. An example was given. This example was new in the sense that all previous examples had involved pairs of uniformities which were not comparable in height.

The question was also raised whether two uniformities exist which are equal in both proximity and height. This question is answered by the example given in this paper.

In §2 we review what we need of proximity and height. In §3 we construct the subbasic covers which are used in §4 to complete the construction of the two uniformities.

2. Proximity and height. The concept of proximity was first introduced by Efremovič in [3] and more about it occurs in [1], [2], and [6]. We say a uniformity $\mathfrak{U}$ is $\leq^p$ to a uniformity $\mathfrak{V}$ if, for any set A in X and any U in $\mathfrak{U}$, there exists V in $\mathfrak{V}$ such that $\mathrm{St}(A, V) \subset \mathrm{St}(A, U)$, where $\mathrm{St}(A, U)$ stands for star of A with respect to U. If both

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$\mathfrak{U} \leqq^p \mathfrak{V}$ and $\mathfrak{V} \leqq^p \mathfrak{U}$, then we say $\mathfrak{U}$ and $\mathfrak{V}$ are in the same proximity class $(=^p)$. For our purposes, the facts needed about proximity are as follows:

2.1. Each proximity class has a unique, smallest, totally bounded member [**1**].

We let $\mathfrak{U}_u$ denote the smallest member of the proximity class of $\mathfrak{U}$. If $\mathfrak{U}$ and $\mathfrak{V}$ are two uniformities, then it follows from a lemma in [**2**] that

2.2. $(\mathfrak{U}\vee\mathfrak{V}_u) =^p (\mathfrak{U}_u\vee\mathfrak{V})$ where $\vee$ denotes least upper bound.

Height was first defined in [**5**]. If $\mathfrak{U}$ and $\mathfrak{V}$ are two uniformities on the same set X, then we say $\mathfrak{U} \leqq^h \mathfrak{V}$ if, for each $U\in\mathfrak{U}$ there exists $V\in\mathfrak{V}$ and a finite cover W of X such that $V\wedge W$ is a refinement of U, where $\wedge$ stands for intersection as in [**6**, p. 4].

From [**5**] we need

2.3. If $\mathfrak{U}$ is a uniformity and $\mathfrak{V}$ is a totally bounded uniformity, then $\mathfrak{V} \leqq^h \mathfrak{U}$.

2.4. If $\mathfrak{U} \leqq^h \mathfrak{V}$ then $(\mathfrak{U}\vee\mathfrak{V}) =^h \mathfrak{V}$.

2.5. If $\mathfrak{U} =^p \mathfrak{V} =^p (\mathfrak{U}\vee\mathfrak{V})$ and $\mathfrak{U} \leqq^h \mathfrak{V}$, then $\mathfrak{U}\leqq\mathfrak{V}$.

2.6. If $\mathfrak{U}$ has a base consisting of partitions, then $\mathfrak{U}_u$ has a base consisting of partitions, $W=\{B_\alpha\}$, $\alpha=1, 2, \cdots, n$, where each B_α is a union of members of some member of the base for $\mathfrak{U}$.

Number 2.6 is a slight generalization of a result appearing in [**1**, p. 354].

3. **Construction of the subbasic covers.** Assume X is a countable set ordered like the integers. Let any subset of X be ordered according to this original ordering. Let f be a one-to-one map of the integers onto the set of all pairs of integers (i, j) where $i<j$, and let π_1 and π_2 be the projections, $\pi_1((i, j))=i$, $\pi_2((i, j))=j$ respectively. Let σ_1 and σ_2 be functions on totally ordered infinite subsets of X defined as follows:

$$\sigma_1(\{x_1, x_2, \cdots\}) = \{x_1, x_3, x_5, \cdots\},$$
$$\sigma_2(\{x_1, x_2, \cdots\}) = \{x_2, x_4, x_6, \cdots\}.$$

Let σ_i^k denote the application of σ_i k times.

We will now inductively construct an infinite number of covers, $\mathfrak{A}_i$, $i=1, 2, \cdots$, of X. First, let $\mathfrak{A}_1=\{A_k^1\}$, $k=1, 2, \cdots$ where $A_k^1=\sigma_1(\sigma_2^{k-1}(X))$. Assume we have constructed $\mathfrak{A}_1, \cdots, \mathfrak{A}_{n-1}$ where $\mathfrak{A}_i=\{A_k^i\}$, and each cover is a partition of X. Also assume that each A_k^i is the union of 2^{i-1} disjoint, infinite sets of the form

3.1. $A_{k_i}^i\cap A_{k_{i-1}}^{i-1}\cap\cdots\cap A_{k_1}^1$ k_r being a member of $f(k_{r+1})$ for $r=1, 2, \cdots, i-1$.

Now, if $l_1=\pi_1(f(k))$ and $l_2=\pi_2(f(k))$, we wish to construct A_k^n so that it intersects each of the 2^{n-2} sets of $A_{l_1}^{n-1}$ and each of the 2^{n-2} sets of $A_{l_2}^{n-1}$ in such a way that the 2^{n-1} resulting sets satisfy the induction hypothesis. Let m_i be the number of times that l_i occurs as a member of $f(j)$ for $j<k$.

Then let A_k^n be the union of the 2^{n-1} sets of the form

$$\sigma_1(\sigma_2^{m_1}(A_{l_2}^{n-1} \cdots)).$$

It is easy to see that these sets satisfy the induction hypothesis and that the sets A_k^n form a partition-cover of X.

Let W^k be a member of the fundamental system of covers for $\mathcal{W}_u^k$ where $\mathcal{W}^k$ is the uniformity generated by $\mathfrak{A}_k$; then we have:

LEMMA 3.2. *For any n and $m<n$,*

3.3. $\mathfrak{A}_n \wedge \mathfrak{A}_{n-1} \wedge \cdots \wedge \mathfrak{A}_{m+1} \wedge W^m \wedge \mathfrak{A}_{m-1} \wedge \cdots \wedge \mathfrak{A}_1$ *does not refine* $\mathfrak{A}_m$.

PROOF. By reference to 3.1 we see that a set of 3.3 must be of the form

3.4. $A_{k_n}^n \cap \cdots \cap A_{k_{m+1}}^{m+1} \cap B_\alpha \cap A_{k_{m-1}}^{m-1} \cap \cdots \cap A_{k_1}^1$, where $B_\alpha \in W^m$.

Now, if it should happen that

3.5. k_{m-1} is a member of both $f(\pi_1(f(k_{m+1})))$ and $f(\pi_2(f(k_{m+1})))$ and

3.6. B_α contains both $A_{\pi_1}^m(f(k_{m+1}))$ and $A_{\pi_2}^m(f(k_{m+1}))$, then 3.2 cannot refine $\mathfrak{A}_m$.

Let $k_{m-1}=1$, for example. Then if $s_1<s_2$ are any two integers such that 1 is a member of both $f(s_1)$ and $f(s_2)$; and s_3 is such that $f(s_3)=(s_1, s_2)$, then we have 3.5, when $k_{m+1}=s_3$ and $k_{m-1}=1$. Clearly, no $B_\alpha \in W$ can contain both $A_{s_1}^m$ and $A_{s_2}^m$, or we would have both 3.5 and 3.6. But there are an infinite number of integers s such that $f(s)$ contains 1. Thus, since W^m has only a finite number of members, 3.5 and 3.6 must occur for some sets in 3.3. Thus, 3.3 cannot refine $\mathfrak{A}_m$.

LEMMA 3.7. *For every $k=1, 2, \cdots$, there exists a two member cover $\beta=\{B_1, B_2\}$ of X such that $\mathfrak{A}_{k+1} \wedge \beta$ refines $\mathfrak{A}_k$.*

PROOF. Let

$$B_i = \bigcup_{j=1}^{\infty} A_j^{k+1} \cap A_{\pi_i(f(j))}^k, \qquad i = 1, 2.$$

4. **Construction of the two uniformities.** Let $\mathcal{U}'$ be generated by the odd numbered $\mathfrak{A}_i$'s and let $\mathcal{V}'$ be generated by the even numbered $\mathfrak{A}_i$'s. Then $\mathcal{U}' =^h \mathcal{V}'$ is a corollary of Lemma 3.7. Let $\mathcal{U}=\mathcal{U}' \vee \mathcal{V}_u'$, and $\mathcal{V}=\mathcal{V}' \vee \mathcal{U}_u'$. By 2.2, $\mathcal{U} =^p \mathcal{V}$. By 2.3 and 2.4, $\mathcal{U} =^h \mathcal{V}$. That

$\mathfrak{U} \neq \mathfrak{V}$ may be seen as follows: Let $\mathfrak{A}_m$ be any member of the generating set for $\mathfrak{U}'$. Now, any member of $\mathfrak{V}$ must be refined by a cover of the form of 3.3 which, by Lemma 3.2, cannot refine $\mathfrak{A}_m$. Thus $\mathfrak{A}_m \notin \mathfrak{V}$.

REMARK. By 2.4, $\mathfrak{U} =^h \mathfrak{U} \vee \mathfrak{V} =^h \mathfrak{V}$. Also $\mathfrak{U} \neq^p \mathfrak{U} \vee \mathfrak{V} \neq^p \mathfrak{V}$, by 2.5. On the other hand, since every proximity class has a smallest member, the greatest lower bound $\mathfrak{U} \overline{\wedge} \mathfrak{V}$ of $\mathfrak{U}$ and $\mathfrak{V}$ is in the same proximity class as both $\mathfrak{U}$ and $\mathfrak{V}$.

If $\mathfrak{U} =^h \mathfrak{U} \overline{\wedge} \mathfrak{V} =^h \mathfrak{V}$, then a simple application of 2.5 proves $\mathfrak{U} = \mathfrak{V}$. Thus, $\mathfrak{U} \neq^h \mathfrak{U} \overline{\wedge} \mathfrak{V} \neq^h \mathfrak{V}$.

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