

ON THE MEASURE OF WELL-DISTRIBUTED SEQUENCES

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0. Introduction. A sequence (x_n) of numbers is said to be uniformly distributed modulo 1 [henceforth we will just write u.d.] if for every subinterval I of the unit interval the following relation holds:

$$(*) \quad \lim_{N \rightarrow \infty} \frac{1}{N} (\# \text{ of } x_n \text{ in } I \text{ (modulo 1), with } 1 \leq n \leq N) = \|I\|$$

where $\|I\|$ is the length of I . It is immediate that if $(*)$ holds then so does

$$(**) \quad \lim_{N \rightarrow \infty} \frac{1}{N} (\# \text{ of } x_n \text{ in } I \text{ (modulo 1), } k \leq n \leq N + k - 1) = \|I\|$$

for each positive integer k . If $(**)$ holds uniformly in k for each subinterval I of the unit interval, then (x_n) is said to be well distributed modulo 1 (w.d.).

Recently there have been several articles (see [3], [5], [6], [11]–[14]) concerned with the following question: suppose (r_n) is a fixed increasing sequence of numbers. What can be said about the measure of the set \mathfrak{W} of x such that the sequence $(r_n x)$ is w.d.? If $\inf(r_n - r_{n-1}) > 0$, it follows by a result of Koksma [10] that the sequences $(r_n x)$ are u.d. for almost all x (in the sense of Lebesgue measure). The sequences (nx) are w.d. for all irrational x ; on the other hand, if a is an integer then it is easy to see that $(a^n x)$ can never be w.d. This last observation has led a number of authors to investigate lacunary sequences (r_n) , i.e., sequences of positive numbers with the property that

$$\liminf_{n \rightarrow \infty} (r_n / r_{n-1}) > 1.$$

In particular, under some additional conditions on the (r_n) it has been shown that $\mu(\mathfrak{W}) = 0$ (where μ is Lebesgue measure). This is the case, for instance, if the ratios r_n / r_{n-1} are integral, or if the r_n are integers and the ratio r_n / r_{n-1} tends to infinity. It is also true if $r_n = \alpha^n$, α a real number. These results have led to the conjecture that for any lacunary sequence (r_n) we have $\mu(\mathfrak{W}) = 0$. It is the purpose of this paper to prove this conjecture and to discuss some other examples.

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We will also note that these results are valid in certain topological groups.

1. The main result.

THEOREM 1. *Let (r_n) be a lacunary sequence and let \mathfrak{W} be the set of x for which the sequence $(r_n x)$ is w.d. Then $\mu(\mathfrak{W}) = 0$.*

PROOF. We will first prove the following lemma from which the theorem will follow easily.

LEMMA 2. *Suppose that, for $n \geq 2$, $r_n/r_{n-1} \geq r > 2$ and that I is any closed subinterval of the unit interval for which $r\|I\| \geq 2$. Then, given any integer k , for almost all x there exists an integer $n (=n(x))$ such that each of the terms $r_{n+1}x, r_{n+2}x, \dots, r_{n+k}x$ lies in I (modulo 1).*

PROOF. Let \emptyset be any bounded interval and let S_k be the above set. We will show that $\mu(\emptyset \cap S_k) \geq \mu(I)^k (1/3)^k \mu(\emptyset)$ which in turn implies, by a well known result, that $\mu(-S_k) = 0$. First of all, notice that there exists r_m such that $\|r_m \emptyset\| \geq 2$. Hence there are disjoint subintervals I_1, \dots, I_p of \emptyset such that

- (i) $r_m I_j \equiv I \pmod{1}$,
- (ii) $(\mu \cup_j I_j) \geq (1/3)\mu(I)\mu(\emptyset)$, (where if $U = [a, b]$ is an interval, then $\alpha U = [\alpha a, \alpha b]$). Since $\mu(I)r \geq 2$ it follows that $\|r_{m+1} I_j\| \geq 2$, so each I_j contains disjoint subintervals I_j^1, \dots, I_j^p ($p = p(j)$) such that

- (i) $r_{m+1} I_j^i \equiv I \pmod{1}$,
- (ii) $\mu(\cup_i I_j^i) \geq (1/3)\mu(I_j)\mu(I)$, so that

$$\mu\left(\bigcup_{i,j} I_j^i\right) \geq (1/3)^2 \mu(I)^2 \mu(\emptyset).$$

Continuing this process by induction our assertion follows.

PROOF OF THE THEOREM. Without loss of generality we may assume that $\inf_{n \geq 2} (r_n/r_{n-1}) \geq r > 1$. There exists an integer N such that $(1/2)(1/N)r^N > 2$. By the lemma, for almost all x and any integer k there exists $n = n(x)$ such that

$$r_{N(n+1)}x, r_{N(n+2)}x, \dots, r_{N(n+k)}x$$

each lie in the interval $[0, (1/2N)] = \mathfrak{U}$ (modulo 1). However, if $(r_n x)$ is w.d. there exists n_0 : the fraction of the number of terms from $r_p x, \dots, r_{p+m} x$ which lie in \mathfrak{U} differs from $1/2N$ by at most $1/(4N)$ for all $m \geq n_0$ and all $p \geq 1$. However, if x is as above, then for $k \geq n_0$ and $n = n(x)$ at least k of the $Nk - N + 1$ terms $r_{Nn+N}x, \dots, r_{Nn+Nk}x$ lie in \mathfrak{U} , a fraction $> 1/N$ and $1/N - 1/(2N) = 1/(2N) > 1/(4N)$, so $(r_n x)$ is not well distributed and the theorem follows.

2. **Some further results.** If $p(x)$ is any nonconstant polynomial, then the sequence $(p(n)x)$ is w.d. except for at most a countably infinite number of x . (For a proof of this result, see [9].) This contrasts with the case $(a^n x)$ (a integral) which, as mentioned before, is never w.d. and raises the question as to further information in both the lacunary case and nonlacunary case. In the lacunary case we have the following

THEOREM 3. *Suppose $\lim(r_n/r_{n-1}) = r > 1$ (where r may be infinite). Then (i) if $r = \infty$, the Hausdorff dimension of \mathfrak{W} is 1.*

(ii) if r is finite, \mathfrak{W} may be empty or (uncountably) infinite.

PROOF. (i) Let (x_n) be any w.d. sequence. Using the same method as Erdős and Taylor [16] we may show that the set of x for which $r_n x \rightarrow x_n$ (modulo 1) has dimension 1, and it is known that if (x_n) is w.d. and $y_n \rightarrow x_n$ (modulo 1) then (y_n) is w.d. so the result follows.

(ii) An example satisfying the first part has already been given. For the second part, let (n_k) be any "rapidly increasing" sequence of integers (say, (n_k) lacunary) and let $\theta = \sum_k (1/2)^{n_k}$. Then it is easy to see, since θ is irrational and the nonzero terms in the binary expansion of θ are "rare," that the sequence $((2^n + n)\theta)$ is w.d.

Notice that the set of θ given in part (ii) has Hausdorff dimension 0. It would be interesting to know whether a set of positive dimension (dimension 1) can be constructed.

In the other direction we have the following

THEOREM 4. *There exists an increasing sequence of positive integers (n_k) , with $1 \leq n_{k+1} - n_k \leq M$ (for some constant M), such that the sequence $(n_k x)$ is not w.d. for any x .*

PROOF. Consider the interval $I = [1/5, 1/4]$. Notice that if none of the terms $nx, (n+1)x, \dots, (n+4)x$ lies in the interval $\mathfrak{U} = [1/8, 3/8] \pmod{1}$ (for $x \in I$), then $(n+5)x$ lies in $I \pmod{1}$. We now construct a sequence of positive integers (n_k) as follows:

Choose $n_1, n_2, 1 \leq n_2 - n_1 \leq 6$, such that $n_i(1/5) \in \mathfrak{U} \pmod{1}, i = 1, 2$. Notice that we may take $n_2 \leq 12$ and that if $|x - 1/5| < 1/(8 \cdot 12)$ then $n_i x \in \mathfrak{O} = [0, 1/2] \pmod{1}$, i.e., if

$$1/5 \leq x \leq 1/5 + 1/(8 \cdot 12).$$

Next, find $n_3, n_4: 1 \leq n_3 - n_2 \leq 6, 1 \leq n_4 - n_3 \leq 6$, such that $n_i v \in \mathfrak{U} (i = 3, 4)$, where $v = 1/5 + 1/(8 \cdot 12)$. Then, $n_4 \leq 4 \cdot 6$ and if $|y - v| \leq 1/(8 \cdot 24)$ then $n_i y \in \mathfrak{O} \pmod{1}, i = 3, 4$. Continue this process. At the k th step we obtain integers n_{2k-1}, n_{2k} such that

(i) $1 \leq n_{2k-1} - n_{2k-2} \leq 6,$

(ii) $1 \leq n_{2k} - n_{2k-1} \leq 6,$

(iii) $n_i z \in \mathfrak{U}, i = 2k - 1, 2k$, where $z = 1/5 + (1/(8 \cdot 12))S$, where $S = 1 + \frac{1}{2} + \dots + 1/(k - 1)$. Hence, if

$$1/5 + S \leq x \leq 1/5 + S + (1/8 \cdot 12)(1/k)$$

then $n_j x \in \mathfrak{O} \pmod{1}, j = 2k - 1, 2k$. Since $\sum 1/n$ diverges we may cover the interval I in this manner, using only finitely many integers n_i , say $i = 1, \dots, m$.

We next find integers $n_{m+1}, n_{m+2}, n_{m+3}$, with gaps as before, such that $n_i(1/5) \in \mathfrak{U}, i = m + 1, m + 2, m + 3$. Again, $n_{m+3} \leq 6(m + 3)$ and if $|x - 1/5| \leq 1/(8 \cdot 6(m + 3))$ then $n_i x \in \mathfrak{O} \pmod{1}, i = m + 1, m + 2, m + 3$. We repeat this process, eventually covering the interval I . At that next stage, we repeat the construction, this time for four consecutive terms lying in $\mathfrak{U} \pmod{1}$. We finally obtain a sequence $(n_k), 1 \leq n_k - n_{k-1} \leq 6$, with the property that if $x \in I$ and if n is any positive integer, then there exists an integer p such that $n_i x \in \mathfrak{O} \pmod{1}$ for $i = p, p + 1, \dots, p + m$. It is clear that if $x \in I$, then $(n_i x)$ is not w.d. Let z be any real number. If z is rational then $(n_i z)$ is not u.d. If z is irrational, then some integral multiple of z , say sz , lies in $I \pmod{1}$, so that the sequence $(n_i sz)$ is not w.d. But, from Weyl's criterion for w.d. sequences it then follows that $(n_i z)$ is not w.d. and the theorem is proved.

3. Distribution in compact abelian groups. Let G be any compact abelian group, ν normalized Haar measure. We can define the notions of a u.d. and a w.d. sequence as before, with open sets replacing intervals and Haar measure replacing Lebesgue measure; equivalently we may require

$$\lim \sum f(x_k) = \int f d\nu$$

for every $f \in C(G)$, or even only for every continuous character χ . Suppose now that G is second countable (so that the character group X is countable) and that G is connected (so that the character group X is torsion-free (see [8, Theorem 24.25])) and let χ be any character, $\chi \neq 1, T$ the unit circle. Then $\chi(G) = T, \chi^{-1}$ is measurable and, if U is any measurable subset of T then $\mu(U) = \nu(\chi^{-1}(U))$ since $\nu\chi^{-1}$ is a translation-invariant measure on T . We can therefore deduce some distribution theorems on G from the corresponding theorems on T . For example, if (n_k) is a strictly increasing sequence of positive integers, then for ν -a.a. x in G the sequence of (x^{n_k}) is u.d. If (n_k) is a lacunary sequence of integers, then for ν -a.a. x the sequence (x^{n_k}) is not w.d. The first-mentioned result generalizes a result of Halmos and

Samelson [7] and Eckmann [4] and could also be proved directly by imitating Koksma's proof [10] using the fact that X is torsion-free.

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