A better result is known,[6] but it is interesting to notice that this can be proved by elementary means since (5) can.[7]

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SOME PROPERTIES OF INTEGRAL CLOSURE

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Let D be an integrally closed domain with identity having quotient field K, let L be an algebraic extension field of K, and let $\overline{D}$ be the integral closure of D in L. We prove here that the following five ideal theoretic structure properties of $\overline{D}$ are inherited by D, namely: (a) $\overline{D}$ is a Prüfer domain, (b) $\overline{D}$ is an almost Dedekind domain,[1] (c) $\overline{D}$ is a Dedekind domain, (d) $\overline{D}$ has the QR-property,[2] (e) $\overline{D}$ has property (#).[3]

The converse of (a) is true (that is, D Prüfer implies that $\overline{D}$ is Prüfer) and was established by Prüfer in [**11**, p. 31]. In case L is finite-dimensional over K, Noether [**9**, p. 37] proved the converse of (c) and Butts and Phillips [**2**, p. 270] proved the converse of (b). In the general case it is well-known or easy to see that the converses of (b), (c) and (e) are false. The converse of (d) is false (see [**6**, p. 102]) even when L is finite-dimensional over K.

Our statements concerning (a), (b) and (c) will be obtained as corollaries to the following.

THEOREM 1. *Let M be a prime ideal of $\overline{D}$ and let $P=M\cap D$. Then $\overline{D}_M\cap K=D_P$.*

PROOF. It is clear that D_P is contained in $\overline{D}_M\cap K$. To obtain the reverse inclusion we first observe that L may be assumed to be a normal extension of K. For let E be a normal closure of L over K, D^* the integral closure of $\overline{D}$ in E, and N a prime ideal of D^* lying over M. Then $N\cap D=P$ and $\overline{D}_M\subseteq D^*_N\cap L$ so that $\overline{D}_M\cap K\subseteq D^*_N\cap K$. Let $\{M_\alpha\}$ be the set of prime ideals of $\overline{D}$ lying over P. By a well-known

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[1] D is *almost Dedekind* if for each maximal ideal P of D, D_P is a discrete rank one valuation ring [**4**, p. 813].

[2] D has the *QR-property* if each integral domain between D and its quotient field is a quotient ring of D [**6**, p. 97].

[3] D is said to have *property* (#) if for Δ_1 and Δ_2 distinct subsets of the set of maximal ideals of $\overline{D}$ we have $\bigcap_{M\in\Delta_1}D_M\neq\bigcap_{M\in\Delta_2}D_M$ [**5**, p. 331].

theorem due originally to Krull [7, p. 752] the M_α's are conjugate over K, that is, for each α there is an automorphism σ_α of L over K such that $\sigma_\alpha(M)=M_\alpha$. It follows that $\sigma_\alpha(\overline{D}_M)=\overline{D}_{M_\alpha}$. Hence $\overline{D}_M\cap K =\overline{D}_{M_\alpha}\cap K=\bigcap_\alpha (\overline{D}_{M_\alpha}\cap K)$. If $S=D-P$ then $\overline{D}_S$ is integral over D_S so that $\overline{D}_S\cap K=D_S$. Moreover, $\{M_\alpha\}$ is the set of ideals of $\overline{D}$ maximal with respect to not meeting S [8, p. 30]. Hence $\{M_\alpha\overline{D}_S\}$ is the set of maximal ideals of $\overline{D}_S$. Since $\overline{D}_{M_\alpha}=(\overline{D}_S)_{M_\alpha\overline{D}_S}$ for each α, we have $\overline{D}_S=\bigcap_\alpha (\overline{D}_S)_{M\alpha\overline{D}_{S_\alpha}}=\bigcap_\alpha \overline{D}_{M_\alpha}$. Therefore, $\overline{D}_M\cap K=\bigcap_\alpha(\overline{D}_{M_\alpha}\cap K)=(\bigcap_\alpha\overline{D}_{M_\alpha})\cap K=\overline{D}_S\cap K=D_S$, as we wished to show.

COROLLARY 1. *With notation as in Theorem* 1 *if $\overline{D}_M$ is a valuation ring, then D_P is a valuation ring.*

COROLLARY 2. *If $\overline{D}_M$ is a valuation ring, then $\overline{D}_{M_\alpha}$ is a valuation ring for each prime ideal M_α of $\overline{D}$ which lies over P.*

PROOF.[4] By Corollary 1, D_P is a valuation ring. If $S=D-P$, then $\overline{D}_S$ is the integral closure of D_S in L so that $\overline{D}_S$ is a Prüfer domain. $\overline{D}_{M_\alpha}$ is a quasi-local overring of $\overline{D}_S$ and thus a valuation ring.

COROLLARY 3. *If $\overline{D}$ is a Prüfer domain, then D is a Prüfer domain.*

COROLLARY 4. *If $\overline{D}$ is an almost Dedekind domain, then D is almost Dedekind.*

PROOF. Let P be a maximal ideal of D and let M be a maximal ideal of $\overline{D}$ lying over P. Since $\overline{D}$ is almost Dedekind, $\overline{D}_M$ is a rank one discrete valuation ring. By Theorem 1, $\overline{D}_M\cap K=D_P$ so that D_P is also a rank one discrete valuation ring. Hence D is almost Dedekind.

COROLLARY 5. *If $\overline{D}$ is a Dedekind domain, then D is Dedekind.*

PROOF.[5] $\overline{D}$ Dedekind implies, using Corollary 4, that D is an almost Dedekind domain in which each nonzero element is contained in only finitely many maximal ideals. Hence by [4, p. 815] D is a Dedekind domain.

A domain which has the QR-property is necessarily Prüfer. Pendleton in [10, p. 500] has shown that a Prüfer domain D has the QR-property if and only if the radical of each finitely generated ideal of D is the radical of a principal ideal. We use this characterization of domains with the QR-property to establish the following.

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[4] If L is normal over K, then the proof of Corollary 2 is immediate; for $\overline{D}_{M_\alpha}$ is then the image of $\overline{D}_M$ under a K-automorphism of L.

[5] Corollary 5 can also be obtained from the fact that an integral domain with identity is Dedekind if and only if it is a Krull domain of dimension ≤ 1.

THEOREM 2. *If $\overline{D}$ is a domain with the QR-property, then D also has the QR-property.*

PROOF. Since $\overline{D}$ has the QR-property, $\overline{D}$ is a Prüfer domain. Hence by Corollary 3, D is a Prüfer domain. Let A be a finitely generated ideal of D. We show that there is some $a \in D$ such that $\sqrt{A} = \sqrt{(a)}$. Denote by $\{P_\alpha\}$ the set of minimal primes of A in D, and by $\{M_\beta\}$ the set of prime ideals of $\overline{D}$ which lie over the P_α's. It follows from the "going down" theorem (see [1, p. 255] or [7, p. 755]) that $\{M_\beta\}$ is the set of minimal primes of $A\overline{D}$. Since $\overline{D}$ has the QR-property and $A\overline{D}$ is finitely generated, there exists $a_1 \in \overline{D}$ such that $\sqrt{(a_1)} = \sqrt{A\overline{D}} = \bigcap_\beta M_\beta$. Hence (a_1) and $A\overline{D}$ are contained in the same prime ideals of $\overline{D}$. Thus $\{M_\beta\}$ is the set of minimal primes of (a_1). Let E be a normal closure of L over K, and let $a_1, a_2, \cdots, a_s$ be the distinct conjugates of a_1 in E. Denote by σ_i a relative isomorphism of L over K into E such that $\sigma_i(a_1) = a_i$. Let $\sigma_i(L) = L_i$ and $\sigma_i(\overline{D}) = \overline{D}_i$. Since σ_i is the identity on D, each $\overline{D}_i$ is an integral extension of D and $\{\sigma_i(M_\beta)\}$ is the set of prime ideals of $\overline{D}_i$ which lie over $\{P_\alpha\}$. And because $\{M_\beta\}$ is the set of minimal primes of (a_1) in $\overline{D}$, $\{\sigma_i(M_\beta)\}$ is the set of minimal primes of (a_i) in $\overline{D}_i$. Denote by p the characteristic of K if K has nonzero characteristic, $p=1$ if K is of characteristic zero. We can choose a positive integer e such that $a = (a_1a_2 \cdots a_s)^{p^e} \in K$. Then $a \in K$ and a integral over D imply $a \in D$. We show that $\{P_\alpha\}$ is the set of minimal primes of aD.

Let D^* be the integral closure of D in E. D^* is the integral closure of $\overline{D}_i$ in E for each i, and $D^* \cap L_i = \overline{D}_i$. If $\{N_\gamma\}$ is the set of primes of D^* which lie over the P_α's, then $N_\gamma \cap \overline{D}_i$ is a prime ideal of $\overline{D}_i$ which lies over some P_α. Hence there is an M_β such that $N_\gamma \cap \overline{D}_i = \sigma_i(M_\beta)$. Also D^* integral over $\overline{D}_i$ implies that for each $\sigma_i(M_\beta)$ there is a prime ideal N of D^* lying over $\sigma_i(M_\beta)$ and $N \cap D = \sigma_i(M_\beta) \cap D = P_\alpha$ for some α. Thus $N \in \{N_\gamma\}$. Because $\{\sigma_i(M_\beta)\}$ is the set of minimal primes of $a_i\overline{D}_i$, $\{N_\gamma\}$ is the set of minimal primes of a_iD^*, $i = 1, 2, \cdots, s$. Hence $\{N_\gamma\}$ is the set of minimal primes of $aD^* = (a_1a_2 \cdots a_s)^{p^e}D^*$. Since $D^* \cap K = D$ and $a \in D$, we have $aD^* \cap D = aD$. It follows that $\{N_\gamma \cap D\} = \{P_\alpha\}$ is the set of minimal primes of $aD^* \cap D = aD$. We have established that $\sqrt{A} = \bigcap_\alpha P_\alpha = \sqrt{aD}$. Hence the radical of each finitely generated ideal of D is the radical of a principal ideal. Therefore D has the QR-property.

We remark that because the QR-property is not preserved under integral extension, the proof of Theorem 2 can not be reduced to the case when L is normal over K.

THEOREM 3. *If $\overline{D}$ has property (#), then D has property (#).*

Proof. We assume that D does not have property (#). Then as observed in [5, p. 331] there is a proper subset $\{P_\alpha\}$ of the set of maximal ideals of D such that $D=\bigcap_\alpha D_{P_\alpha}$. Let $\{M_\beta\}$ be the set of prime ideals of $\overline{D}$ lying over the P_α's. Since $\overline{D}$ is integral over D, $\{M_\beta\}$ is a proper subset of the set of maximal ideals of $\overline{D}$. We show that $\overline{D}=\bigcap_\beta \overline{D}_{M_\beta}$ and hence that $\overline{D}$ does not have property (#). Let $u\in L-\overline{D}$ and let $f(X)\in K[X]$ be the minimal polynomial for u over K. Since u is not integral over D, $f(X)\notin D[X]$. Because $D=\bigcap_\alpha D_{P_\alpha}$, there is a $P\in\{P_\alpha\}$ such that $f(X)\notin D_P[X]$. If $S=D-P$, then $\overline{D}_S$ is the integral closure of D_S in L and $f(X)\notin D_S[X]$ implies that u is not integral over D_S. Hence $u\notin\overline{D}_S$. If $\{M_\gamma\}$ is the set of prime ideals of $\overline{D}$ which lie over P, then $\{M_\gamma\}\subseteq\{M_\beta\}$ and the maximal ideals of $\overline{D}_S$ are of the form $M_\gamma\overline{D}_S$. We have $\overline{D}_S=\bigcap_\gamma (\overline{D}_S)_{M_\gamma\overline{D}_S}=\bigcap_\gamma \overline{D}_{M_\gamma}$. Therefore $u\notin\overline{D}_{M_\gamma}$ for some γ so that $u\notin\bigcap_\beta \overline{D}_{M_\beta}$. It follows that $\overline{D}=\bigcap_\beta \overline{D}_{M_\beta}$ which completes the proof of Theorem 3.

If J is an integral domain having quotient field F, we call any domain between J and F an *overring* of J. Using Theorem 3 and the fact that an integral domain J is Prüfer if and only if each overring of J is integrally closed [3, p. 198] we obtain the following corollary.

Corollary 6. *If every overring of $\overline{D}$ has property (#), then every integrally closed overring of D has property (#). In particular, if $\overline{D}$ is a Prüfer domain and every overring of $\overline{D}$ has property (#), then every overring of D has property (#).*

In connection with Corollary 6 the following question arises: If J^*, the integral closure of a domain J has property (#), must J have property (#)? We have been unable to answer this question.

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