SCHAUDER DECOMPOSITIONS IN (m)

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1. **Introduction.** If the Banach space B is a countably infinite direct sum of nontrivial subspaces B_i, $B=\sum B_i$, such that each b has a unique representation $b=\sum b_i$, $b_i \in B_i$; and if each of the projections $P_i b = b_i$ is continuous then say that B has a Schauder decomposition (a brief history of this topic is found in [8]). Sanders [9] has shown that (m), the space of bounded sequences, does not have a Schauder decomposition such that the coordinate sequences $(\delta_{ij})_{j=1}^{\infty}$ lie in distinct B_k. The purpose of this note is to prove (m) has no Schauder decomposition whatsoever. In fact this is true for a class of spaces which include those $C(H)$ spaces for which H is compact, Hausdorff, and extremely disconnected; (and all direct factors of such spaces (the P_λ-spaces) (see [1, pp. 94–96] for a discussion of such $C(H)$).

Let B^* be the conjugate space of B and let (f_n) be a sequence in B^*. If $\lim_n f_n(b)$ exists and is finite for every b in B then the limit defines an element of B^* [2, p. 52]. The following properties of (m) are critical for our proof.

(1) If $\lim f_n(b) = f(b)$ exists for each b in (m) then $\lim F(f_n)$ exists for each F in $(m)^{**}$ and is equal to $F(f)$, [3, pp. 168, 169]. That is, sequential weak* convergence in $(m)^*$ implies weak convergence.

(2) Let X, Y be Banach spaces and let S, T be weakly compact linear operators such that

$$X \xrightarrow{S} (m) \xrightarrow{T} Y.$$

Then $TS\colon X \to Y$ is compact [2, p. 494].

Replacing (m) with B in (1) and (2) our main theorem is: If B has properties (1) and (2) then it does not have a Schauder decomposition.

2. **The main theorem.** Suppose now that B has property (1) and a Schauder decomposition $\sum B_j$. Then $b=\sum_1^\infty P_j(b)$ for each b in B. Define conjugate projections Q_j in B^* by $Q_j f(b) = f(P_j b)$ for every b in B, f in B^*. Each sum $\sum_n^m P_j$ is a continuous projection and has conjugate projection $\sum_n^m Q_j$. The set of projections $\sum_n^m P_j$ is uniformly norm bounded (in m, n) since $(\sum_n^m P_j)(b) = \sum_n^m b_j \to 0$, $n, m \to \infty$ [2, p. 52]. As $\|\sum_n^m Q_j\| = \|\sum_n^m P_j\|$ the set of projections

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$\sum_n^m Q_j$ is uniformly norm bounded as well. Moreover for each f in B^* one has

$$\left(\sum_n^m Q_j f\right)(b) = \sum_n^m f(P_j b) = f\left(\sum_n^m P_j b\right) \to 0, \qquad n, m \to \infty.$$

Then, using (1), $\sum_1^\infty Q_j f$ converges weakly to f. It is easy to see that the sequence $(Q_j f)$ is unique. Thus, in this sense, B^* has a weak decomposition $B^* = \sum_1^\infty B_j'$, where $B_j' = Q_j B^*$. In fact $\sum B_i'$ is a Schauder decomposition of B^* ((i, ii) below).

PROPOSITION. *If B has a Schauder decomposition and satisfies* (1) *then*

(i) *If $b_j \in B_j$ for each j and either $b_j = 0$ or $\|b_j\| = 1$ then $\{b_j \mid b_j \neq 0\}$ is a basis for $[b_j]$ (the closed span of (b_j)).*

(ii) *If $f_j \in B_j'$ for each j and either $f_j = 0$ or $\|f_j\| = 1$ then $\{f_j \mid f_j \neq 0\}$ is a basis for $[f_j]$.*

(iii) *$\sum B_j'$ is a Schauder decomposition of B^*.*

(iv) *$[f_j]$ in* (ii) *is reflexive.*

(v) *Let $F \in B^{**}$ $(=(B^*)^*)$ and let F_n be F restricted to $\sum_n^\infty B_j'$. Then $\|F_n\| \to 0$ (the decomposition is shrinking in the sense of* [8]).

(vi) *$[b_j]$ in* (i) *above is reflexive.*

Parts (iv) and (vi) are those needed to prove the main theorem, the others being steps to achieving (iv) and (vi).

Suppose now the proposition is known and suppose B satisfies (1), (2), and has Schauder decomposition $\sum B_j$. Choose $b_j \in B_j$ with $\|b_j\| = 1$. Let $B^* = \sum B_j'$ be the decomposition in (iii). Choose c such that $\|Q_j\| \leq c$ for every j. Select g_j having norm one such that $|g_j(b_j)| \geq 1/2$. Then $|Q_j g_j(b_j)| \geq 1/2$ while $\|Q_j g_j\| \leq c$. Set $f_j = Q_j g_j / \|Q_j g_j\|$. Then $1 = \|f_j\| \geq |f_j(b_j)| \geq 1/2c > 0$. Find such f_j in B_j' for each j. Recall that if $i \neq j$ then $f_j(b_i) = 0$ $(Q_j f_j(P_i b_i) = f_j(P_j P_i b_i) = f_j(0))$. Consider the mapping

$$[b_j] \xrightarrow{I} B \xrightarrow{T} [f_j]^*$$

where I is the identity and T is defined by $Tb(\sum t_j f_j) = \sum t_j f_j(b)$. Then I and T are continuous and linear. Since by (iv) and (vi) both $[b_j]$ and $[f_j]$ are reflexive $[f_j]^*$ is also, and so I and T are weakly compact, [1, p. 56]. Thus by (2) TI is compact. Now $TIb_j(f_j) = Tb_j(f_j) = f_j(b_j)$ so $\|TIb_j\| \geq 1/2c$. A subsequence TIb_{n_j} converges to an element of $[f_j]^*$ while TIb_{n_j} converges weakly to 0 since

$$TIb_{n_j}\left(\sum_1^\infty t_i f_i\right) = \sum_1^\infty t_i f_i(b_{n_j}) = t_{n_j} f_{n_j}(b_{n_j}) \xrightarrow{j} 0$$

and in a reflexive space weak and weak* convergence are the same [**1**, p. 25]. Thus TIb_{n_j} converges in norm to 0 which is a contradiction to $\|TIb_j\| \geqq 1/2c$ for all j.

3. **Proof of the proposition.** We shall need two well known theorems about bases, the first due to James [**4**], and the second to Karlin [**5**] (see [**1**, p. 69] for a proof). A basis (b_n) for B is boundedly complete if $\|\sum_1^n t_j b_j\| \leqq K$ for all n implies $\sum_1^\infty t_j b_j$ converges. It is shrinking if, for each f in B^* $f_n = f|[b_j, j \geqq n]$ (f restricted to the closed span of $\{b_j | j \geqq n\}$) converges to 0 ($\|f_n\| \to 0$).

(J) Let B have basis (b_n). Then B is reflexive if and only if (b_n) is shrinking and boundedly complete.

A sequence (b_n) is a weak Schauder basis if for each b in B there is a unique sequence (t_j) such that $\sum t_j b_j$ converges weakly to b and if the functionals $\phi_i(b) = t_i$ are continuous.

(K) A weak Schauder basis is a basis.

Both (i) and (ii) will follow from (iii) and its proof. To show (iii), that is that $\sum B_j'$ is a Schauder decomposition, we must show $f = \sum Q_j f$, for every f in B^*. Now $\sum_1^n Q_j f$ converges weakly* to f and so weakly to f. Thus each f has a unique representation $(Q_j f)$ such that $\sum_1^n Q_j f$ converges weakly to f and the coordinate projection Q_j is continuous for each j. Such a decomposition we may call a weak decomposition. The following lemma was observed first by Ruckle [**7**, Theorem 1.20 and a remark on p. 549].

LEMMA. *If $X = \sum_1^\infty X_j$ is a weak decomposition then it is a Schauder decomposition.*

PROOF. Let $\sum_1^\infty x_j$ converge weakly to x, $x_j \in X_j$ for each j. Let $y_j = x_j/\|x_j\|$ if $x_j \neq 0$ and $y_j = 0$ if $x_j = 0$. In $[y_j]$ let $z_n = \sum_{j=1}^\infty t_{jn} y_j$ converge to u as $n \to \infty$. Then, since each projection $R_j x = x_j$ is continuous n: $t_{jn} y_j \to u_j$ so that t_{jn} converges, say to t_j, if $y_j \neq 0$ (and if $y_j = 0$ we may set each $t_{jn} = 0$). Moreover $u_j = t_j y_j$. The sequence (t_j) is unique and $\sum_1^n t_j y_j = \sum_1^n u_j$ converges weakly to u. Thus (y_j) is a weak basis for $[y_j]$ and by (K) it is a basis (this proves (i) then).

Now the representation of x in $[y_j]$ is $\sum_1^\infty \|x_j\| y_j = \sum_1^\infty x_j$ which converges to x as (y_j) is a basis. Thus the decomposition is a Schauder decomposition (hence (ii) follows from (i) in our particular case). We remark that the argument proves such a sequence (y_j) is a basis for $[y_j]$ for any space X having a Schauder decomposition $\sum X_j$.

(iv) To show $[f_j]$ in (ii) is reflexive we show its unit ball is weakly compact [**1**, pp. 51, 56]. Let $\sum_{i=1}^\infty t_{jn} f_j = g_n \in [f_j]$ and $\|g_n\| \leqq 1$ for each n. Use a diagonal process to find a subsequence $g_{n_i} = h_i = \sum s_{ji} f_j$ such that $s_{ji} \to s_j$ as $i \to \infty$ for each j (the projections (Q_j) are uniformly norm bounded so this may be done).

By (iii) $\sum_1^\infty Q_j$ converges to the identity in the strong operator topology [1, p. 34] so that the set $(\sum_m^\infty Q_j)$ is uniformly norm bounded, say by c. Then for each x in B

$$\begin{aligned} \left| h_n(x) - h_m(x) \right| &\leqq \left| \sum_1^k (s_{jn} f_j - s_{jm} f_j)(x_j) \right| + \left| \sum_{k+1}^\infty (s_{jn} f_j - s_{jm} f_j)(x_j) \right| \\ &\leqq \left| \sum_1^k (s_{jn} f_j - s_{jm} f_j)(x_j) \right| \\ &\quad + 2c \left| \sum_{k+1}^\infty x_j \right| < \left| \sum_1^k (s_{jn} f_j - s_{jm} f_j)(x_j) \right| + \epsilon \end{aligned}$$

if K is large, $<2\epsilon$ if n, m are large. Thus h_n is weakly* Cauchy and so weakly Cauchy. Since $h_n(x_i) \to s_i f_i(x_i)$ as $n \to \infty$ for every x_i in B_i one has that $h_n(x) \to \sum s_i f_i(x)$ for every x in B. But then h_n converges weakly to $\sum s_i f_i$ in B^*. Since $\sum s_i f_i$ is in $[f_j]$ the unit ball of $[f_j]$ is weakly compact.

(v) Let F be in B^{**} and $F_n = F \mid \sum_n B_j'$. Suppose $\|F_n\| \not\to 0$. Then $\exists \epsilon > 0$ and a subsequence (F_{n_i}) of (F_n) such that $\|F_{n_i}\| > \epsilon$ for each i. If $n_i > m$ then $F_m = F_{n_i}$ on the domain of F_{n_i} so that $\|F_m\| \geqq \|F_{n_i}\|$. Hence $\|F_n\| > \epsilon$ for every n. Choose $f_1 = \sum_1^{n_1} g_j$ such that $1 \geqq \|f_1\| \geqq 1/2$ and $F_1(f_1) > \epsilon$. Choose $f_2 = \sum_{n_1+1}^{n_2} g_j$ such that $1 \geqq \|f_2\| \geqq 1/2$ and $F_{n_1+1}(f_2) > \epsilon$. Continuing in this way we choose a sequence F_{n_j+1} and $f_j = \sum_{n_{j-1}+1}^{n_j} g_j$ such that $1 \geqq \|f_j\| \geqq 1/2$ and $F_{n_{j-1}+1}(f_j) > \epsilon$ $(n_0 = 0)$. Now $[g_j]$ is reflexive so by (J) $F \mid [g_i \mid i \geqq n_j]$ converges to 0 in norm. But $F(f_j) = F_{n_{j-1}+1}(f_j) > \epsilon$ for each j contradicts $f_j \in [g_j \mid j \geqq n_{j-1} + 1]$ for large j.

(vi) To show $[b_j]$ in (i) is reflexive we prove that $[b_j]^*$ has a basis which is both shrinking and boundedly complete and apply (J). Choose f_j in B_j such that $f_j(b_j) = 1$. The identity I: $[b_j] \to B$ has conjugate I^* such that $I^* f_j = \phi_j$ is a biorthogonal sequence to (b_j). Given any ϕ in $[b_j]^*$ there is a unique sequence (t_j) such that $\phi(x) = \sum t_j \phi_j(x)$ for each x in $[b_j]$ (let $\phi(b_j) = t_j$, $x = \sum s_j b_j$, then $\phi(x) = \sum t_j s_j = \sum t_j \phi_j(x)$). Let $I^* g = \phi$ and $g = \sum g_j$. Then $I^* g_j = t_j \phi_j$ for some t_j since $I^* g_j(\sum s_i b_i) = s_j g_j(b_j) = g_j(b_j)\phi_j(\sum s_i b_i)$. Thus $t_j = g_j(b_j)$. Thus $I^*(\sum g_j) = \sum t_j \phi_j$ converges. By the calculation preceding, it must converge to ϕ so that (ϕ_j) is a basis for $[b_j]^*$. If $\|\sum_1^n t_j \phi_j\| \leqq K$ for each n then

$$\sum_n^m t_j \phi_j\left(\sum s_i b_i\right) \xrightarrow{nm} 0$$

or $\sum_1^n t_j\phi_j$ converges weakly* to a limit ϕ and $\phi(b_j)=t_j$. Thus $\phi=\sum t_j\phi_j$ so that

$$\sum_1^n t_j\phi_j \xrightarrow{n} \sum t_j\phi_j.$$

Thus (ϕ_j) is boundedly complete and it remains to show it is shrinking. Let $F\in[\phi_j]^*$ and let $F_n(\sum t_j\phi_j)=F(\sum_n^\infty t_j\phi_j)$ for every $\sum t_j\phi_j$ in $[\phi_j]$. Then $\|F_n\|\to 0$ if and only if $\|F|\,[\phi_j|j\geqq n]\|\to 0$ as follows. The projections $\sum_1^\infty t_j\phi_j\to\sum_n^\infty t_j\phi_j$ are uniformly norm bounded, say by K. Then $\|F|\,[\phi_j|j\geqq n]\|\leqq\|F_n\|\leqq K\|F|\,[\phi_j|j\geqq n]\|$. Now $I^{**}F_n=I^{**}F$ on $\sum_n^\infty B_j'$ and 0 on $\sum_1^{n-1}B_j'$. Then, as above, $\|I^{**}F_n\|\to 0$ if $\|I^{**}F|\sum_n^\infty B_j'\|\to 0$ which is (v). Since $\|I^{**}F_n\|=\|F_n\|$ (I^{**} is an isometry) the basis is shrinking.

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