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THE MOMENTS OF RECURRENCE TIME

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In connection with Poincaré's recurrence theorem Kac [1] obtained the mean of the recurrence time (formula (3) below) and the author [2] gave a very simple proof of this result. Recently Blum and Rosenblatt [3] obtained[2] the higher moments (formula (2) below). In the present note we obtain both results by an exceedingly simple and perspicuous argument. This note is entirely self-contained.

Let Ω be a point set, m a probability measure on Ω, and T a one-to-one ergodic measure-preserving transformation of Ω into itself. Let $A\subset\Omega$ be such that $m(A)>0$. For any point a in Ω let $n(a)$ be the smallest positive integer such that $T^n a\in A$; if no such integer exists let $n(a)=\infty$. Define $A_k=\{a\in A\mid n(a)=k\}$, $\bar{A}=\Omega-A$, and $\Gamma_k=\{a\in\bar{A}\mid n(a)=k\}$. Borrowing the notation of [3] we will define

$$p_n = m\{\Gamma_n \cup \Gamma_{n+1} \cup \cdots\}, \tag{1}$$

for $n\geqq 1$. We will also make use of the usual combinatorial symbol $(k)_j=k(k-1)\cdots(k-j+1)$ for k and j positive integers, with $(k)_0=1$.

Our object will be to prove that

$$D_j = \int_A [n(a)]_j dm = j(j-1)\sum_{k=j-2}^{\infty} (k)_{(j-2)} p_{k+1} \tag{2}$$

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[2] These moments have also been obtained by F. H. Simons, Notice #40 of the Eindhoven Technical School, December 23, 1966.

for $j \geqq 2$, the result of [3]. The result of [1] (also proved in [2]) is

$$D_1 = 1. \tag{3}$$

By Poincaré's recurrence theorem (e.g., [2]; ergodicity of T is not required) one has that $m(A_\infty)=0$. The ergodicity of T implies that $m(\Gamma_\infty)=0$.

The basic formula of our argument will be

$$T(A_k \cup \Gamma_k) = \Gamma_{k-1} \tag{4}$$

for $k \geqq 2$; it is so obvious as not to require proof. Using (4) repeatedly for $k=n+1, n+2, \cdots$ we obtain that

$$m(\Gamma_n) = \sum_{k=n+1}^{\infty} m(A_k), \qquad n \geqq 1; \tag{5}$$

$$p_n = \sum_{k=n+1}^{\infty} (k-n)m(A_k), \qquad n \geqq 1. \tag{6}$$

Thus

$$\begin{aligned} p_1 &= m(A_2) + 2m(A_3) + 3m(A_4) + \cdots \\ &= D_1 - \sum_{k=1}^{\infty} m(A_k) = D_1 - m(A). \end{aligned} \tag{7}$$

Obviously

$$p_1 = m(\overline{A}) = 1 - m(A), \tag{8}$$

so that (7) and (8) prove (3).

Using (6) in the right member of (2) we obtain that the coefficient of $m(A_k)$, $k \geqq j$, in the right member of (2) is

$$j(j-1)\left[\sum_{i=1}^{k-j+1} i(k-i-1)_{(j-2)}\right], \tag{9}$$

which is easily shown (e.g., by induction) to equal $(k)_j$. This proves (2).

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