NONUNIQUENESS OF EXTREMAL KERNELS[1]

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Given a "kernel" $k = k(e^{i\theta}) \in L^q$, then

$$\Phi(f) = \frac{1}{2\pi i}\int_{|z|=1} f(z)k(z)\,dz$$

is a bounded linear functional on H^p $(1/p+1/q=1)$ with norm $\|\Phi\| \leqq \|k\|_q$. Two kernels induce the same linear functional, $k_1 \sim k_2$, if and only if they differ by an H^q function. It can be shown that the duality relation, $\|\Phi\| = \inf\{\|k_1\|_q : k_1 \sim k_2\}$ holds for $1 \leqq p \leqq \infty$. One may ask whether an extremal kernel exists, i.e., a kernel $k_1 \sim k$ such that $\|\Phi\| = \|k_1\|_q$, and if so, whether it is unique.

By functional analytic methods, Havinson [1] and Rogosinski and Shapiro [2] showed that an extremal kernel always exists for $1 \leqq p \leqq \infty$ and is unique for $1 < p \leqq \infty$. However, Rogosinski and Shapiro constructed a counterexample to show that it need not be unique for $p=1$. As their example to show nonuniqueness for $p=1$ was rather complicated, we present a simplification of their example.

Let

$$\begin{aligned} k(e^{i\theta}) &= 1, && 0 \leqq \theta < \pi/2,\\ &= -1, && \pi/2 \leqq \theta < \pi,\\ &= 0, && \pi \leqq \theta < 2\pi. \end{aligned}$$

Rogosinski and Shapiro showed that this kernel induces a functional Φ on H^1 with $\|\Phi\| = 1$, and thus k is an extremal kernel. To show this extremal kernel is not unique, it suffices to produce an H^∞ function $h \neq 0$ for which $\|k+h\|_\infty = 1$.

Let h be the conformal mapping sending the unit disk to the upper half-disk $|w| < 1$, $\operatorname{Im} w > 0$, with $h(1) = -1$, $h(i) = 0$, and $h(-1) = 1$. Then $h \in H^\infty$, and

$$\begin{aligned} -1 \leqq h(e^{i\theta}) &< 0, && 0 \leqq \theta < \pi/2\\ 0 \leqq h(e^{i\theta}) &< 1, && \pi/2 \leqq \theta < \pi\\ |h(e^{i\theta})| &= 1, && \pi \leqq \theta < 2\pi. \end{aligned}$$

Thus $|k(e^{i\theta}) + h(e^{i\theta})| \leqq 1$, and h is the desired function.

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References

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POSITIVE $H^{1/2}$ FUNCTIONS ARE CONSTANTS

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The Koebe function $z/(1+z)^2$ is positive everywhere on $|z|=1$, $z\neq -1$, and lies in the Hardy class H^p for every $p<1/2$. We show that this behavior is extreme by proving the following

THEOREM. *If* $f(z)\in H^{1/2}$ *and if* $f(z)\geqq 0$ *a.e. on* $|z|=1$ *then* $f(z)$ *is a constant.*

PROOF. We may assume that $f(z)$ is not identically 0. If $B(z)$ denotes the Blaschke product for the zeros of $f(z)$ then, as usual, we can write

$$f(z) = B(z)F^2(z), \qquad F(z)\in H^1. \tag{1}$$

We write the condition $f(z)\geqq 0$ as the equation $f(z)=|f(z)|$ and conclude from (1) that

$$B(z)F^2(z) = |F^2(z)| \quad \text{a.e. on } |z| = 1. \tag{2}$$

Since $f(z)$ is not identically 0 it follows that $F(z)$ is nonzero a.e. on $|z|=1$. Thus we may divide (2) by $F(z)$ and obtain

$$B(z)F(z) = \overline{F(z)} \quad \text{a.e. on } |z| = 1. \tag{3}$$

But the left side of (3) is H^1 and so has all negative Fourier coefficients 0, the right side is conjugate H^1 and so has all positive Fourier coefficients 0!.

Thus only the constant term remains and we conclude that both sides are constants. This is to say $B(z)F(z)$ and $F(z)$ are both constants and so indeed $f(z)=(B(z)F(z))\cdot F(z)$ is a constant.

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