ON ODD PERFECT NUMBERS. III

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It is not known whether or not odd perfect numbers exist. However, many interesting necessary conditions for an odd integer to be perfect have been found out. A bibliography of previous work on odd perfect numbers is given by McCarthy [**3**].

Throughout this paper n denotes an odd perfect number. The following results have been proved in [**4**], [**6**] and [**5**] respectively:

(i) $\prod_{p|n} p/(p-1) < (175/96)\,\zeta(3) < 2.19125$.

(ii) n is of the form $12t+1$ or $36t+9$.

(iii) If n is of the form $36t+9$ and $5|n$, then

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log\left(\frac{65}{61}\right) \qquad (\sim 0.674).$$

(iv) If n is of the form $36t+9$ and $5\nmid n$, then

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{338} + \log\left(\frac{18}{13}\right) \qquad (\sim 0.662).$$

The object of this paper is to improve the upper bound for the product $\prod_{p|n} p/(p-1)$ given by (i) above. We prove the following:

THEOREM. (α) *If n is of the form $12t+1$ and $5|n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{56791}{33612}\cdot\zeta(3) < 2.031002.$$

(β) *If n is of the form $12t+1$ and $5\nmid n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{1760521}{1050375}\cdot\zeta(3) < 2.014754.$$

(γ) *If n is of the form $36t+9$ and $5|n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{318897}{177023}\cdot\zeta(3) < 2.165439.$$

(δ) *If n is of the form $36t+9$ and $5\nmid n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{3706148208}{2125240975}\cdot\zeta(3) < 2.096234.$$

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PROOF. Euler proved that n must be of the form $p_0^{\alpha_0}\cdot x^2$, where p_0 is a prime of the form $4\lambda+1$, α_0 is of the form $4\mu+1$, $x>1$ and $(p_0, x)=1$. Hence we can write $n=p_0^{\alpha_0}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where α_r is even for $1\leqq r\leqq k$. We shall suppose without loss of generality that $p_1<p_2<\cdots<p_k$. Let $\sigma(n)$ denote the sum of all the positive divisors of n. Since n is a perfect number, we have $\sigma(n)=2n$, from which it can easily be seen that

$$\text{(A)}\qquad \prod_{r=0}^{k}\frac{p_r}{p_r-1}=2\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1} > 2.$$

Throughout the following q_r denotes the rth prime, counting 2 as the first prime. We make use of the following well-known identity due to Euler:

$$\text{(B)}\qquad \prod_{r=1}^{\infty}(1-q_r^{-3})^{-1}=\zeta(3),$$

where $\zeta(s)$ is the Riemann Zeta function.

(a) Suppose n is of the form $12t+1$. In this case, it has been proved in [4, p. 134] that p_0 is of the form $12N+1$ and hence $p_0\geqq 13$.

(a_1) Suppose $5|n$ and $7|n$. Then $p_1=5$, $p_2=7$. Now $\alpha_2\geqq 4$. For, if $\alpha_2=2$, then $\sigma(p_2^{\alpha_2})=3.19$ and since $\sigma(n)=2n$, it would follow that $3|n$, which can not hold.

($a_{1.1}$) If $p_0=13$, then $p_3\geqq 11$ and $p_r\geqq q_{r+3}$ for $4\leqq r\leqq k$. Since α_r is even for $1\leqq r\leqq k$, $\alpha_2\geqq 4$ and $\alpha_0\geqq 1$, we have

$$\begin{aligned}\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}&<(1-13^{-2})^{-1}(1-5^{-3})^{-1}(1-7^{-5})^{-1}(1-11^{-3})^{-1}\\&\qquad\times\prod_{r=4}^{k}(1-p_r^{-3})^{-1}\\&<(1-13^{-2})^{-1}(1-5^{-3})^{-1}(1-7^{-5})^{-1}(1-11^{-3})^{-1}\\&\qquad\times\prod_{r=7}^{\infty}(1-q_r^{-3})^{-1}\\&=\frac{(1-2^{-3})(1-3^{-3})(1-7^{-3})(1-13^{-3})}{(1-13^{-2})(1-7^{-5})}\cdot\zeta(3),\qquad\text{by (B)}\\&=\frac{56791}{67224}\cdot\zeta(3).\end{aligned}$$

($a_{1.2}$) If $p_0 \neq 13$, then since p_0 is of the form $12N+1$, $p_0 \geqq 37$. $p_r \geqq q_{r+2}$ for $3 \leqq r \leqq k$. Hence

$$\prod_{r=0}^{k} (1-p_r^{-(\alpha_r+1)})^{-1} < (1-37^{-2})^{-1}(1-5^{-3})^{-1}(1-7^{-5})^{-1} \prod_{r=5}^{\infty} (1-q_r^{-3})^{-1}$$

$$< \frac{56791}{67224} \cdot \zeta(3).$$

Hence, by (A), (α) follows in the case (a_1).

(a_2) Suppose $5 | n$ and $7 \nmid n$, then $p_1 = 5$ and $p_r \geqq q_{r+3}$ for $2 \leqq r \leqq k$. Since α_0 is odd, $(1+p_0) | \sigma(p_0^{\alpha_0})$ and hence $\{(1+p_0)/2\} | n$, since $\sigma(n) = 2n$. Now, $p_0 \neq 13$. For, otherwise, by the above, it would follow that $7 | n$, which is not the case. Since p_0 is of the form $12N+1$, $p_0 \geqq 37$. Hence

$$\prod_{r=0}^{k} (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1}(1 - 5^{-3})^{-1} \prod_{r=5}^{\infty} (1 - q_r^{-3})^{-1}$$

$$< \frac{56791}{67224} \cdot \zeta(3).$$

Hence, by (A), (α) follows in this case also. Thus (α) is proved.

(a_3) Suppose $5 \nmid n$ and $7 | n$, then $p_1 = 7$. Now, $\alpha_1 \geqq 4$. For, if $\alpha_1 = 2$, it would follow as in (a_1) that $3 | n$, which does not hold.

($a_{3.1}$) If $p_0 = 13$, then $p_2 \geqq 11$ and $p_r \geqq q_{r+4}$ for $3 \leqq r \leqq k$. Hence

$$\prod_{r=0}^{k} (1-p_r^{-(\alpha_r+1)})^{-1} < (1-13^{-2})^{-1}(1-7^{-5})^{-1}(1-11^{-3})^{-1} \prod_{r=7}^{\infty} (1-q_r^{-3})^{-1}$$

$$= \frac{(1-2^{-3})(1-3^{-3})(1-5^{-3})(1-7^{-3})(1-13^{-3})}{(1-13^{-2})(1-7^{-5})} \cdot \zeta(3)$$

$$= \frac{1760521}{2100750} \cdot \zeta(3).$$

($a_{3.2}$) If $p_0 \neq 13$, then $p_0 \geqq 37$ and $p_r \geqq q_{r+3}$ for $2 \leqq r \leqq k$. Hence

$$\prod_{r=0}^{k} (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1}(1 - 7^{-5})^{-1} \prod_{r=5}^{\infty} (1 - q_r^{-3})^{-1}$$

$$< \frac{1760521}{2100750} \cdot \zeta(3).$$

Hence, by (A), (β) follows in case (a_3).

(a_4) If $5\nmid n$ and $7\nmid n$, then $p_r \geqq q_{r+4}$ for $1\leqq r\leqq k$. As in (a_2), $p_0\neq 13$ and hence $p_0\geqq 37$. Hence

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-37^{-2})^{-1}\prod_{r=5}^{\infty}(1-q_r^{-3})^{-1}$$
$$<\frac{1760521}{2100750}\cdot\zeta(3).$$

Hence, by (A), (β) follows in this case also. Thus (β) is proved.

(b) Suppose n is of the form $36t+9$. Since $3|n$, $p_1=3$.

(b_1) If $5|n$, then $7\nmid n$ in virtue of the result that $3\cdot 5\cdot 7$ does not divide n (proved by Kuhnel, p. 203 of [2]).

($b_{1.1}$) Suppose $p_0=5$.

($b_{1.1.1}$) If $11|n$, then $\alpha_0=1$ in virtue of the result that $3\cdot 5^2\cdot 11$ does not divide n (proved by Kanold [1, p. 26]). In this case $p_2=11$ and $p_r\geqq q_{r+3}$ for $3\leqq r\leqq k$. Further, $\alpha_2\geqq 4$. For, if $\alpha_2=2$, then $\sigma(p_2^{\alpha_2})=133=7.19$ and since $\sigma(n)=2n$, it would follow that $7|n$, which is not the case. Also, $\alpha_1\geqq 4$. For, if $\alpha_1=2$, then $\sigma(p_1^{\alpha_1})=\sigma(3^2)=13|n$ and this implies that

$$\sum_{p|n}\frac{1}{p}>\frac{1}{3}+\frac{1}{5}+\frac{1}{11}+\frac{1}{13}>\frac{1}{3}+\frac{1}{5}+\frac{1}{13}+\log\left(\frac{65}{61}\right),$$

a contradiction to (iii). Hence

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-5^{-2})^{-1}(1-3^{-5})^{-1}(1-11^{-5})^{-1}\prod_{r=6}^{\infty}(1-q_r^{-3})^{-1}$$
$$<\frac{318897}{354046}\cdot\zeta(3).$$

Hence, by (A), (γ) follows in this case.

($b_{1.1.2}$) Suppose $11\nmid n$. Then $\alpha_1\neq 4$. For, if $\alpha_1=4$, then $\sigma(p_1^{\alpha_1})=121=11^2$ and since $\sigma(n)=2n$, it would follow that $11|n$, which is not the case.

Hence, either $\alpha_1=2$ or $\alpha_1\geqq 6$.

Suppose $\alpha_1=2$. Then $\sigma(p_1^{\alpha_1})=13|n$ and in this case both 17 and 19 together do not divide n. For, otherwise, it would follow that

$$\sum_{p|n}\frac{1}{p}>\frac{1}{3}+\frac{1}{5}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}>\frac{1}{3}+\frac{1}{5}+\frac{1}{13}+\log\left(\frac{65}{61}\right),$$

a contradiction to (iii). Hence

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-5^{-2})^{-1}(1-3^{-3})^{-1}(1-13^{-3})^{-1}(1-17^{-3})^{-1}$$
$$\times\prod_{r=9}^{\infty}(1-q_r^{-3})^{-1},$$

if $17|n$, $19\nmid n$; and

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-5^{-2})^{-1}(1-3^{-3})^{-1}(1-13^{-3})^{-1}(1-19^{-3})^{-1}$$
$$\times\prod_{r=9}^{\infty}(1-q_r^{-3})^{-1},$$

if $17\nmid n$, $19|n$; and

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-5^{-2})^{-1}(1-3^{-3})^{-1}(1-13^{-3})^{-1}\prod_{r=9}^{\infty}(1-q_r^{-3})^{-1},$$

if $17\nmid n$, $19\nmid n$.

Since $1<(1-19^{-3})^{-1}<(1-17^{-3})^{-1}$, it follows that in all the three cases, we have

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-5^{-2})^{-1}(1-3^{-3})^{-1}(1-13^{-3})^{-1}(1-17^{-3})^{-1}$$
$$\times\prod_{r=9}^{\infty}(1-q_r^{-3})^{-1}$$
$$=\frac{318897}{354046}\cdot\zeta(3).$$

If $\alpha_1\geqq 6$, then

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-5^{-2})^{-1}(1-3^{-7})^{-1}\prod_{r=6}^{\infty}(1-q_r^{-3})^{-1}$$
$$<\frac{318897}{354046}\cdot\zeta(3).$$

Hence, by (A), (γ) follows in the case ($b_{1.1.2}$) also.

($b_{1.2}$) Suppose $p_0\neq 5$. Then $p_2=5$. In this case, as in (a_2), $p_0\neq 13$. For, otherwise, it would follow that $7|n$, which can not hold. Since p_0 is of the form $4\lambda+1$, $p_0\geqq 17$. $p_r\geqq q_{r+2}$ for $3\leqq r\leqq k$. Hence

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-17^{-2})^{-1}(1-3^{-3})^{-1}(1-5^{-3})^{-1}\prod_{r=5}^{\infty}(1-q_r^{-3})^{-1}$$

$$<\frac{318897}{354046}\cdot\zeta(3).$$

Hence, by (A), (γ) follows in this case also. Thus (γ) is proved.

(b_2) Suppose $5\nmid n$. Since p_0 is of the form $4\lambda+1$, $p_0\geqq 13$. Also, $p_0\neq 29$. For, otherwise, it would follow as in (a_2) that $(1+p_0)/2=3\cdot 5\mid n$, and this implies that $5\mid n$, which is not the case.

($b_{2.1}$) If $7\mid n$, then $p_2=7$.

($b_{2.1.1}$) Suppose $\alpha_1=\alpha_2=2$. Then both 13 and 19 divide n, since $\sigma(p_1^{\alpha_1})=13$, $\sigma(p_2^{\alpha_2})=57=3\cdot 19$ and $\sigma(n)=2n$. In this case, neither 11 nor 17 divides n. For, otherwise, it would follow that

$$\sum_{p\mid n}\frac{1}{p}>\frac{1}{3}+\frac{1}{7}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}>\frac{1}{3}+\frac{1}{338}+\log\left(\frac{18}{13}\right),$$

a contradiction to (iv). Hence, either (1) $p_0=13$, $p_3=19$, $p_r\geqq q_{r+5}$ for $4\leqq r\leqq k$; or (2) $p_0\neq 13$, $p_3=13$, $p_4=19$, $p_r\geqq q_{r+4}$ for $5\leqq r\leqq k$. In the second case, $p_0\geqq 37$, since p_0 is of the form $4\lambda+1$, $p_0\neq 13$, $17\nmid n$ and $p_0\neq 29$. In the first case,

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-13^{-2})^{-1}(1-3^{-3})^{-1}(1-7^{-3})^{-1}(1-19^{-3})^{-1}$$
$$\times\prod_{r=9}^{\infty}(1-q_r^{-3})^{-1}$$
$$=\frac{1853074104}{2125240975}\cdot\zeta(3).$$

In the second case,

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-37^{-2})^{-1}(1-3^{-3})^{-1}(1-7^{-3})^{-1}(1-13^{-3})^{-1}$$
$$\times(1-19^{-3})^{-1}\prod_{r=9}^{\infty}(1-q_r^{-3})^{-1}$$
$$<\frac{1853074104}{2125240975}\cdot\zeta(3).$$

($b_{2.1.2}$) If at least one of α_1 and α_2 is not equal to 2, then either $\alpha_1\geqq 4$, $\alpha_2\geqq 4$ or $\alpha_1\geqq 4$, $\alpha_2=2$ or $\alpha_1=2$, $\alpha_2\geqq 4$. The proofs for the first two cases are omitted as they are similar to the previous proofs. In both these cases, we easily verify that the upper bound obtained for

$\prod_{p/n} p/(p-1)$ is less than the bound obtained in the third case. In the third case, we have

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}$$
$$<(1-13^{-2})^{-1}(1-3^{-3})^{-1}(1-7^{-5})^{-1}\prod_{r=5}^{\infty}(1-q_r^{-3})^{-1}$$
$$<\frac{1853074104}{2125240975}\cdot\zeta(3).$$

Hence, by (A), (δ) follows in any case under ($b_{2.1}$).

($b_{2.2}$) If $7\nmid n$, then $p_r \geqq q_{r+3}$ for $2\leqq r\leqq k$. Hence

$$\prod_{r=0}^{k}(1-p_r^{-(\alpha_r+1)})^{-1}<(1-3^{-3})^{-1}\prod_{r=5}^{\infty}(1-q_r^{-3})^{-1}$$
$$<\frac{1853074104}{2125240975}\cdot\zeta(3).$$

Hence, by (A), (δ) follows in this case also. Thus (δ) is proved. Thus the proof of the theorem is complete.

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