PRIME $(-1, 1)$ RINGS WITH IDEMPOTENT[1]

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1. **Introduction.** Right alternative rings arise when the alternative identity is weakened [1]. That is, a ring R is called right alternative if the identity $(y, x, x)=0$ is satisfied for all x, y in R where the associator is defined as $(x, y, z)=(xy)z-x(yz)$. When the characteristic of R is prime to 2 this is equivalent to the identity

$$(1) \qquad (x, y, z) + (x, z, y) = 0 \qquad \text{for all } x, y, z \in R.$$

Many authors have investigated right alternative rings (see the bibliography). In this paper we examine a subclass of these rings, the $(-1, 1)$ rings. Such rings R satisfy (1) and the identity

$$(2) \qquad (x, y, z) + (y, z, x) + (z, x, y) = 0 \qquad \text{for all } x, y, z \in R.$$

Maneri [7] proved that a simple ring of type $(-1, 1)$ with characteristic prime to 6 having an idempotent $e\neq 0, 1$ is associative. It is shown in this paper that when R is a $(-1, 1)$ ring with no trivial ideals which has characteristic prime to 6, then if R contains an idempotent $e\neq 0, 1$, it has a Peirce decomposition relative to e. Further, the multiplicative relations between the submodules of the Peirce decomposition relative to containment are the same as those for an associative ring. Under the additional assumption that R is a prime ring it is proven that R must be associative.

2. **Preliminary section. A Peirce decomposition.** We will assume throughout this section that R is a $(-1, 1)$ ring with characteristic prime to 6 having an idempotent $e\neq 0, 1$. When other conditions on R are needed they will be noted.

The commutator is defined as $(x, y)=xy-yx$ where $x, y\in R$. It is simple to verify that the identity $C(x, y, z)=(xy, z)-x(y, z)-(x, z)y-(x, y, z)+(x, z, y)-(z, x, y)=0$ is satisfied by all elements x, y, z in an arbitrary ring. When R satisfies (1) this reduces to

$$(3) \quad C(x, y, z) = (xy, z) - x(y, z) - (x, z)y - 2(x, y, z) - (z, x, y) = 0.$$

The following identities hold in an arbitrary right alternative ring [5].

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(4) $J(x, w, y, z) = (x, w, yz) + (x, y, wz) - (x, w, z)y - (x, y, z)w = 0,$

(5) $K(x, y, z) = (x, y^2, z) - (x, y, yz + zy) = 0.$

When R, in addition, satisfies (2) and has an idempotent e, the following identities are satisfied for all $x, y, w \in R$ [7].

(6) $$(x, (e, e, y)) = 0,$$

(7) $$(e, e, y)(e, e, w) = 0,$$

(8) $$(e, e, (e, x))y = (e, e, y(e, x)).$$

Next, define $U = \{u \in R \mid (u, R) = 0\}$. Then if u is in U, $0 = C(x, x, u) = -2(x, x, u)$ and so

(9) $$(x, x, u) = 0.$$

From (1) it then follows that $(x, u, x) = 0$. If x is replaced by $x+y$ in (9) and the last equation we obtain

(10) $$(x, y, u) = -(y, x, u), \qquad (x, u, y) = -(y, u, x).$$

We will find the Teichmüller identity, which follows, useful. For all $x, y, z, w \in R$, where R is an arbitrary ring,

(11) $$\begin{aligned} F(x, y, z, w) = (xy, z, w) - (x, yz, w) + (x, y, zw) \\ - x(y, z, w) - (x, y, z)w = 0. \end{aligned}$$

LEMMA 1. *Let $A \neq 0$ be an ideal of R. Then the set of two-sided annihilators of A is an ideal of R.*

PROOF. Suppose x is in R and $xA = Ax = 0$. Let a belong to A and y to R. From (1) we get $0 = (y, a, x) + (y, x, a) = (y, x, a) = (yx)a$ and $0 = (x, a, y) + (x, y, a) = (x, y, a) = (xy)a$. Then from (2), $0 = (a, x, y) + (x, y, a) + (y, a, x) = (a, x, y) + (x, y, a) = -a(xy) + (xy)a = -a(xy)$ and $0 = (a, y, x) + (y, x, a) + (x, a, y) = (a, y, x) + (y, x, a) = -a(yx) + (yx)a = -a(yx)$. Hence, the two-sided annihilators of A form an ideal of R.

The next lemma is crucial to the existence of a Peirce decomposition relative to e.

LEMMA 2. *Let R have characteristic prime to 6. Then let s belong to R such that s and sR belong to U. Define*

$$B_s = \{x \in R \mid xs = x(sR) = (xR)s = (Rx)s = (xR)(Rs) = (Rx)(Rs) = 0\}.$$

Then B_s is an ideal of R.

PROOF. Let $x \in B_s$ and $y, z, w \in R$. From the definition of B_s, it is immediate that

$$(xw)s = (wx)s = (xw)(ys) = (wx)(ys) = 0. \tag{12}$$

Now $(wx, y, s) = -(y, wx, s)$ and $(xw, y, s) = -(y, xw, s)$ from (10) since s is in U. Expanding these associators, we have $[(wx)y]s = -[y(wx)]s$ and $[(xw)y]s = -[y(xw)]s$. However, $(wx, y, s) = -(wx, s, y) = 0$ and $(xw, y, s) = -(xw, s, y) = 0$ from (12). We conclude that

$$[(wx)y]s = [(xw)y]s = [y(wx)]s = [y(xw)]s = 0. \tag{13}$$

It remains to show that $[(xw)y](sz) = [(wx)y](sz) = [y(xw)](sz) = [y(wx)](sz) = 0$. We will show that the expressions involving xw vanish, and note that an identical proof applies to the remaining two expressions. We have, from (4), $0 = J(xw, s, y, z) = (xw, s, yz) + (xw, y, sz) - (xw, s, z)y - (xw, y, z)s$. Since $(xw, s, yz) = (xw, s, z)y = 0$ by (12) and (13) we obtain $(xw, y, sz) = (xw, y, z)s$. Combining this equation with the fact that sz is in U we get $(xw, y, sz) = -(xw, sz, y) = (y, sz, xw) = (xw, y, z)s$. Expanding the latter three associators and using (12) and (13) we obtain

$$[y(sz)](xw) = (xw)[(sz)y] = \{[(xw)y]z\}s. \tag{14}$$

Next, we have $0 = (xw, y, sz) + (y, sz, xw) + (sz, xw, y) = -2(xw, sz, y) - (sz, y, xw) = 2(xw)[(sz)y] - [(sz)y](xw) + (sz)[y(xw)]$ from (2), (10) and (12). Then, from (14) and the fact that $sz \in U$, we obtain

$$(xw)[(sz)y] = -(sz)[y(xw)]. \tag{15}$$

Now, from (1), $(xw, y, sz) = -(xw, sz, y)$. Expanding these associators and using (12) we obtain $[(xw)y](sz) - (xw)[y(sz)] = (xw)[(sz)y]$. Since sz belongs to U this equation becomes

$$2(xw)[(sz)y] = [(xw)y](sz). \tag{16}$$

Combining (15) and (16) we get

$$2(sz)[y(xw)] + [(xw)y](sz) = 0. \tag{17}$$

From (14) and (16) and the fact that s belongs to U we have

$$2\{[(xw)y]z\}s = [(xw)y](zs). \tag{18}$$

It then follows from (18) that $((xw)y, z, s) = -\{[(xw)y]z\}s$. However, $((xw)y, z, s) = -((xw)y, s, z) = [(xw)y](sz)$ from (13) and so, $[(xw)y](sz) = -\{[(xw)y]z\}s$. This equation, combined with (18), yields $[(xw)y](zs) = 0$ when divided by 3. Whence, from (17) and the fact that sz is in U, it follows that $[y(xw)](sz)=0$. Thus B_s is an ideal of R.

We now assume that R has no trivial ideals. The following lemma leads directly to the existence of the Peirce decomposition.

LEMMA 3. $(e, e,(e, x))=0$ *for all* x *in* R.

PROOF. Let $(e, e, (e, x))=b$. Then from (6) we obtain $(b, R)=0$. Furthermore, $(bR, R)=0$ from (8). Thus the element b satisfies the requirements for the element s of Lemma 2.

On the other hand, it is quite clear from (6), (7), and (8) that b also belongs to B_b. Now let C be the ideal generated by b. Then C is contained in B_b. But from Lemma 2 it is evident that $B_bb=bB_b=0$. Then from Lemma 1 it follows that since the two-sided annihilators of an ideal form an ideal of R, $CB_b=B_bC=0$ and so $C^2=0$. That is, C is a trivial ideal of R. Hence, $C=0$.

THEOREM 1. *Let R be a $(-1, 1)$ ring with no trivial ideals. Further, suppose that the characteristic of R is prime to 6. Then if R has an idempotent, e, R has the desired Peirce decomposition $R=R_{11}+R_{10}+R_{01}+R_{00}$ where x belongs to R_{ij} if and only if $ex=ix$ and $xe=jx$ for $i, j=0, 1$ and the sum of the submodules is direct.*

PROOF. It suffices to show that $(e, e, x)=(e, x, e)=(x, e, e)=0$ for all x in R. From the fact that R is right alternative, $(x, e, e)=0$ for all x in R. Next, from (5), $0=K(e, e, x)=(e, e, x)-(e, e, ex+xe)$. Since $(e, e, (e, x))=0$ from Lemma 3 we obtain $(e, e, x)-2(e, e, ex)=0$. Replacing x with ex in the last equation we get $(e, e, ex)-2(e, e, e(ex))=0$. Since (e, e, x) is in U, it follows from (9) with e and (e, e, x) substituted for x and u that $0=(e, e, (e, e, x))=(e, e, ex)-(e, e, e(ex))$. Thus, $(e, e, ex)=0$. But then $(e, e, x)=0$. Finally, from (1), $(e, x, e)=0$ and R has the desired Peirce decomposition relative to e.

3. **Main section.** Let R be a $(-1, 1)$ ring with no trivial ideals containing an idempotent $e\neq 0, 1$. Under the assumption that the characteristic of R is prime to 6 the following two lemmas are satisfied by R [7].

LEMMA 4. *The following multiplication table, with respect to containment, holds for the submodules R_{ij} of the Peirce decomposition of R.*

$$R_{ij}R_{km}\subseteq \delta_{jk}R_{im} \text{ except } R_{ij}^2\subseteq R_{ii} \text{ where } i, j, k, m = 0, 1 \text{ and } \delta_{jk} \text{ is the Kronecker delta.} \tag{19}$$

LEMMA 5. *For all x, y in R and the idempotent e the following identity holds.*

$$(20) \qquad (e, x, y) = 0.$$

We now proceed to examine the submodules R_{10} and R_{01} carefully. We will show that $R_{ij}^3=0$ when $i=0$, 1 and $j=1-i$ and that R_{ij}^2 is in the center of R. It is then easy to show that R_{ij}^2 is a trivial ideal of R and hence must be zero.

LEMMA 6. *$R_{ij}^3=0$ for $i=0$, 1 and $j=1-i$.*

PROOF. Let x_{ij}, y_{ij}, z_{ij} belong to R_{ij}. Then

$$0=(y_{ij}, x_{ij}, x_{ij})=(y_{ij}x_{ij})\ x_{ij}-y_{ij}(x_{ij}^2).$$

But $y_{ij}x_{ij}=0$ from (19) and so

$$(21) \qquad (y_{ij}x_{ij})x_{ij} = 0.$$

Next, $0=(x_{ij}, y_{ij}, e)+(y_{ij}, e, x_{ij})+(e, x_{ij}, y_{ij})=(x_{ij}, y_{ij}, e)+(y_{ij}, e, x_{ij})$ from (2) and (20). Expanding these associators we get

$$(22) \qquad (x_{ij}, y_{ij}) = 0.$$

Combining (21) and (22) and using (19) we obtain $0=(x_{ij}y_{ij})x_{ij}$ $=(x_{ij}, y_{ij}, x_{ij})=-(x_{ij}, x_{ij}, y_{ij})=-x_{ij}^2y_{ij}$. Then replacing x_{ij} with $x_{ij}+z_{ij}$ in the last expression we get $(x_{ij}z_{ij}+z_{ij}x_{ij})y_{ij}=0$. This result, combined with (22), yields $(x_{ij}z_{ij})y_{ij}=0$. Hence $R_{ij}^3=0$.

LEMMA 7. *$(R, R_{ij}^2)=0$ where $i=0$, 1 and $j=1-i$.*

PROOF. From Lemma 6 and (19), it suffices to consider commutators (x, y) where x belongs to $R_{ji}\cup R_{ii}$ and y to R_{ij}^2. First, let x_{ij}, y_{ij} belong to R_{ij} and z_{ji} to R_{ji}. Now $0=(x_{ij}, y_{ij}, z_{ji})+(x_{ij}, z_{ji}, y_{ij})$ $=(x_{ij}, z_{ji}, y_{ij})$ from (19). Then $0=(x_{ij}, y_{ij}, z_{ji})+(y_{ij}, z_{ji}, x_{ij})$ $+(z_{ji}, x_{ij}, y_{ij})=(z_{ji}, x_{ij}, y_{ij})=-z_{ji}\ (x_{ij}y_{ij})$ from (2) and (19). Hence $R_{ji}R_{ji}^2=R_{ij}^2R_{ji}=0$.

It remains to show that elements from R_{ij}^2 commute with elements of R_{ii}. To this end, let x_{ij}, y_{ij} belong to R_{ij} and z_{ii} to R_{ii}. Then $0=(x_{ij}, y_{ij}, z_{ii})+(x_{ij}, z_{ii}, y_{ij})=(x_{ij}y_{ij})z_{ii}-x_{ij}(z_{ii}y_{ij})$ from (19). Since the elements in R_{ij} commute, (22), we have

$$(23) \qquad (x_{ij}y_{ij})z_{ii} = (z_{ii}y_{ij})x_{ij} = (y_{ij}x_{ij})z_{ii} = (z_{ii}x_{ij})y_{ij}.$$

Next, $0=(z_{ii}, x_{ij}, y_{ij})+(x_{ij}, y_{ij}, z_{ii})+(y_{ij}, z_{ii}, x_{ij})=(z_{ii}, x_{ij}, y_{ij})$ $+(x_{ij}y_{ij})z_{ii}-y_{ij}(z_{ii}x_{ij})$ from (2) and (19). Then from (19), (23) and the fact that elements from R_{ij} commute with each other we conclude that

$$(24) \qquad (z_{ii}, x_{ij}, y_{ij}) = 0.$$

Expanding this associator and using (22) and (23) we get

$0=(z_{ii}x_{ij})y_{ij}-z_{ii}(x_{ij}y_{ij})=(y_{ij}x_{ij})z_{ii}-z_{ii}(x_{ij}y_{ij})$. Hence $(R_{ii}, R_{ij})=0$ and we conclude that $(R, R_{ij}^2)=0$.

To prove that R_{ij}^2 where $i=0$, 1 and $j=1-i$ are in the center of R it remains to show that they are contained in the nucleus of R.

LEMMA 8. *R_{ij}^2 is in the center of R when $i=0$, 1 and $j=1-i$.*

PROOF. Consider associators of the form $(x_{kp}, y_{ij}z_{ij}, w_{mn})$ where x_{kp} belongs to R_{kp}, w_{mn} to R_{mn} and y_{ij}, z_{ij} to R_{ij} and $k, p, m, n=0, 1$.

We recall, first, that $R_{ij}+R_{ji}+R_{jj}$ annihilates R_{ij}^2 from both sides. Hence the above associator vanishes unless $k=p=i$ or $m=n=i$. Suppose that $k=p=i$. If $m=j$, $0=(x_{ii}, y_{ij}z_{ij}, w_{jn})$ from (19) regardless of the value of n. Next, let $k=p=m=i$ and $n=j$. Then $0=(x_{ii}, y_{ij}z_{ij}, w_{ij})+(x_{ii}, w_{ij}, y_{ij}z_{ij})=(x_{ii}, y_{ij}z_{ij}, w_{ij})$ from (19) and Lemma 7.

Elements from R_{ij}^2 belong to U by Lemma 7. Hence, from (10), $(x_{kp}, y_{ij}z_{ij}, w_{mn})=-(w_{mn}, y_{ij}z_{ij}, x_{kp})$. If we now assume that $m=n=i$ the argument of the last paragraph applies and we can conclude that unless $k=p=m=n=i$, the original associator vanishes.

We have left to consider associators of the form $(x_{ii}, y_{ij}z_{ij}, w_{ii})$. From (4) and (24) we obtain $0=J(x_{ii}, w_{ii}, y_{ij}, z_{ij})=(x_{ii}, w_{ii}, y_{ij}z_{ij})+(x_{ii}, y_{ij}, w_{ii}z_{ij})-(x_{ii}, w_{ii}, z_{ij})y_{ij}-(x_{ii}, y_{ij}, z_{ij})w_{ii}=(x_{ii}, w_{ii}, y_{ij}z_{ij})-(x_{ii}, w_{ii}, z_{ij})y_{ij}$. However, $0=(x_{ii}, w_{ii}, z_{ij})+(x_{ii}, z_{ij}, w_{ii})=(x_{ii}, w_{ii}, z_{ij})$ from (19) and so we have $0=(x_{ii}, w_{ii}, y_{ij}z_{ij})=-(x_{ii}, y_{ij}z_{ij}, w_{ii})$. Therefore, R_{ij}^2 is in the middle nucleus of R. It follows from (1) and (2) that whenever an element is contained in the middle nucleus of R it is contained in the nucleus of R. Hence, R_{ij}^2 is in the nucleus of R.

LEMMA 9. *R_{ij}^2 is a trivial ideal of R, where $i=0$, 1 and $j=1-i$.*

PROOF. $(R_{ij}^2, R)=0$ and $(R_{ij}+R_{ji}+R_{jj})R_{ij}^2=0$. Also, from (23), when x_{ij}, y_{ij} belong to R_{ij} and z_{ii} to R_{ii}, $(x_{ij}y_{ij})z_{ii}=(z_{ii}y_{ij})x_{ij}$. Therefore, R_{ij}^2 is an ideal of R. But, from Lemmas 6 and 8, when x_{ij}, y_{ij}, z_{ij}, w_{ij} belong to R_{ij}, $(x_{ij}y_{ij})(z_{ij}w_{ij})=[(x_{ij}y_{ij})z_{ij}]w_{ij}=0$. Hence, R_{ij}^2 is a trivial ideal of R.

COROLLARY. *$R_{ij}^2=0$ where $i=0$, 1 and $j=1-i$.*

PROOF. Immediate from Lemma 9 and the fact that R has no trivial ideals.

It is now clear that the submodules R_{ij}, where $i, j=0, 1$, satisfy the same multiplicative relations as those for an associative ring with idempotent. Namely,

(25) $R_{ij}R_{kp}\subseteq \delta_{jk}R_{ip}$ where $i, j, k, p = 0, 1$ and δ_{jk} is the Kronecker delta.

As an immediate consequence of (25), by direct computation, e is contained in the nucleus of R.

The next lemma is true for arbitrary rings of type $(-1, 1)$.

LEMMA 10. *Let y belong to the nucleus of R where R is a ring of type $(-1, 1)$. Then (y, z) belongs to the nucleus of R for all z in R.*

PROOF. Let x, w, z belong to R. Then from (4), $0=J(x, w, y, z) = (x, w, yz)+(x, y, wz)-(x, w, z)y-(x, y, z)w=(x, w, yz)-(x, w, z)y$. Also, from (11), $0=F(x, w, z, y)=(xw, z, y)-(x, wz, y)+(x, w, zy) -x(w, z, y)-(x, w, z)y=(x, w, zy)-(x, w, z)y$. Combining the above results, we conclude that $(x, w, yz)=(x, w, zy)$ and thus $(x, w, (y, z))=0$. Therefore, (y, z) is in the right nucleus of R. Finally, from (1) and (2), (y, z) is contained in the nucleus of R.

LEMMA 11. *The set $B=R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}$ is an ideal in the nucleus of R.*

PROOF. It was mentioned above that e is in the nucleus of R. Let x_{ij} belong to R_{ij}. Then from Lemma 10, (e, x_{ij}) also belongs to the nucleus of R. But $(e, x_{ij})=\pm x_{ij}$ when $i\neq j$ and so R_{10} and R_{01} are in the nucleus of R. From this fact and (25), it is immediate that B is an ideal of R contained in the nucleus of R.

We now make the additional assumption that R is a prime ring. A ring R is called prime if, whenever I and J are ideals in R such that $IJ=0$, then either $I=0$ or $J=0$.

LEMMA 12. *Let R be an arbitrary nonassociative prime ring. Then R can contain no nonzero nuclear ideals.*

PROOF. Let A be an ideal in the nucleus of R. Then if x, y, z, w belong to R and a to A, we have $0=F(a, x, y, z)=(ax, y, z)-(a, xy, z) +(a, x, yz)-a(x, y, z)-(a, x, y)z=-a(x, y, z)$. Further, $a[(x, y, z)w] =[a(x, y, z)]w=0$. But finite sums of elements of the form (R, R, R) and $(R, R, R)R$ form a 2-sided ideal in an arbitrary ring. Hence A annihilates an ideal of R containing all associators of R. Since R is prime and not-associative, $A=0$.

It is now clear that $B=R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}=0$ and R assumes the form $R=R_{11}+R_{00}$ unless R is associative. But then R_{11} and R_{00} become orthogonal ideals of R. Since R is prime this means that either $R_{11}=0$ or $R_{00}=0$. However, $e\neq 0$ belongs to R_{11}. Thus $R_{00}=0$. If this is the case, $R=R_{11}$ and e is the identity of R, contrary to our assumption that $e\neq 1$. Hence, R must be associative.

THEOREM 2. *Let R be a prime ring of type $(-1, 1)$ with characteristic* *prime to* 6. *If* R *has an idempotent* $e \neq 0$, 1, *then* R *is associative.*

It should be noted that an arbitrary primitive ring is also prime [9]. Hence, by constructing a suitable radical, the results of this paper could be extended to semisimple rings.

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