# APPROXIMATION OF CERTAIN CONTINUOUS FUNCTIONS OF $S^2$ INTO $E^3$

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Bing has shown [1] that a homeomorphism of $S^2$ into $E^3$ may be polyhedrally approximated as close as desired. In this paper, we present sufficient conditions to assure that continuous functions of $S^2$ into $E^3$ may be approximated by homeomorphisms.

The usual metric for $S^2$ and $E^3$ will be denoted by $\rho$.

A disk is the homeomorphic image of $[0, 1]\times[0, 1]=I^2$. A singular disk is the continuous image of $I^2$ which cannot be realized by a homeomorphism. If $A$ is a subset of a singular disk $D$ and $f$ is the associated function from $I^2$ to $D$, $A$ will be called nonsingular if $f^{-1}$ restricted to $A$ is a function.

If $f$ and $g$ are continuous functions from $X$ into $Y$, *the distance from $f$ to $g$, $\rho(f, g)$*, is the least upper bound of the set of all numbers $\rho(f(x), g(x))$ for $x\in X$; $d(f)$ is the greatest lower bound of the set of all numbers $\rho(f, h)$ where $h$ is a homeomorphism of $X$ into $Y$ and indicates how closely $f$ may be approximated by a homeomorphism. If $f$ is a continuous function from $X$ into $Y$, *the set of singular points of $f$, $K(f)$*, is the closure of the set of all $x\in X$ with the property that there is $y\in X$, $y\neq x$, such that $f(y)=f(x)$.

The following will be proven:

THEOREM. *If $f$ is a continuous function from $S^2$ into $E^3$ and*

(i) *$f(K(f))$ is 0-dimensional,*

(ii) *no component of $K(f)$ separates $S^2$,*

*then $d(f)=0$.*

It will be convenient to have a few lemmas as aids in proving the theorem.

LEMMA 1. *Suppose $G_1, G_2, \cdots, G_N$ is a collection of singular disks contained in a 3-manifold with boundary, $M$, of $E^3$ satisfying*

(i) *each $\mathrm{Bd}(G_i)$ is nonsingular, polyhedral, and misses $\mathrm{Int}(G_i)$,*

(ii) *if $i\neq j$, $\mathrm{Bd}(G_i)\cap G_j=\emptyset$.*

*Then there is a collection $G_1', G_2', \cdots, G_n'$ of mutually disjoint polyhedral disks contained in $M$ such that $\mathrm{Bd}(G_i)=\mathrm{Bd}(G_i')$.*

The proof of Lemma 1, obtained from Dehn's Lemma [2] and trading disks, is standard.

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LEMMA 2. *If $f$ is a continuous function from $S^2$ into $E^3$ satisfying*
(i) *$K(f)$ is 0-dimensional,*
(ii) *$f(K(f))$ is 0-dimensional,*
*then $d(f)=0$.*

PROOF. We show that for each $\epsilon>0$, there is a homeomorphism $g$ of $S^2$ into $E^3$ such that $\rho(f, g)<\epsilon$.

Since each point of $f(S^2-K(f))$ is at a positive distance from $f(K(f))$, by Bing's approximation theorem (Theorem 7 of [1]) we may assume that $f(S^2-K(f))$ is locally polyhedral.

Since $f(K(f))$ is 0-dimensional, there is a collection $M_1', \cdots, M_p'$ of mutually disjoint 3-manifolds each of diameter less than $\epsilon$ such that $f(K(f))\subset\sum \mathrm{Int}(M_i')$. Since $K(f)$ is 0-dimensional, there is a collection $D_1, \cdots, D_N$ of mutually disjoint disks in $S^2$ such that $K(f) \subset \sum \mathrm{Int}(D_i)$, $f(\mathrm{Bd}(D_i))$ is polyhedral and $\sum f(D_i) \subset\sum \mathrm{Int}(M_j)$.

So that we may apply Lemma 1, in each $M_J'$ we construct a 3-manifold with boundary, $M_J$, and a continuous function $F$ from $S^2$ into $E^3$ satisfying
(i) $F=f$ on $S^2-\sum \mathrm{Int}(D_i)$,
(ii) $F(\sum \mathrm{Bd}(D_i))\subset\sum \mathrm{Bd}(M_J)$, $F(\sum \mathrm{Int}(D_i))\subset\sum \mathrm{Int}(M_J)$,
(iii) for each $i$, there is a neighborhood of $F(D_i)$ which is nonsingular and polyhedral
(iv) if $i\neq j$, $\mathrm{Bd}(F(D_i))\cap F(D_j)=\varnothing$.

Suppose $D_{J(1)}, \cdots, D_{J(t)}$ are the disks in $S^2$ such that $f(\sum D_{J(i)}) \subset\mathrm{Int}(M_J')$. For each $D_{J(i)}$, there is a disk $E_{J(i)}$ such that $D_{J(i)}\cap K(f) \subset\mathrm{Int}(E_{J(i)})\subset\mathrm{Int}(D_{J(i)})$. Since $f(\sum E_{J(i)})$ is at a positive distance from both $f(S^2-\sum \mathrm{Int}(D_{J(i)}))$ and the complement of $M_J'$, $f(\sum E_{J(i)})$ may be covered by a finite collection $\sigma_1, \cdots, \sigma_r$ of tetrahedrons in general position and $f(\sum E_{J(i)})\subset\sum \mathrm{Int}(\sigma_j)$, $\sum \sigma_j\subset\mathrm{Int}(M_J')$ and $\sum \sigma_j\cap f(S^2-\sum \mathrm{Int}(D_i))=\varnothing$. Each annulus $f(D_{J(i)}-\mathrm{Int}(E_{J(i)}))$ may be covered by a solid torus $T_{J(i)}$ contained in $M_J'$ such that $T_{J(i)}$ is in general position with respect to $\sum \sigma_j$ and each torus $T_{J(k)}$, $k\neq i$, $f(\mathrm{Bd}(D_{J(i)}))\subset\mathrm{Bd}(T_{J(i)})$, $T_{J(i)}\cap f(S^2-\sum D_{J(i)})=\varnothing$, and for $k \neq i$, $T_{J(i)} \cap f(\mathrm{Bd}(D_{J(k)})) = \varnothing$. Since we may assume $\sum T_{J(i)}\cup\sum \sigma_j$ is connected, $M_J=\sum T_{J(i)}\cup\sum \sigma_j$ is the desired 3-manifold with boundary. Define $F$ on $\sum E_{J(i)}$ to be a simplicial approximation to $f$ such that

$$F\left( \sum E_{J(i)}\right) \subset M_J,\ F\left( \sum E_{J(i)}\right) \cap f\left( \sum \mathrm{Bd}(D_{J(i)})\right) = \varnothing$$

and $F=f$ on $\sum(D_{J(i)}-E_{J(i)})$.

The collection $F(\sum D_{J(i)})$ and $M_J$ satisfy the conditions of Lemma 1. Let $g_J$ be the natural homeomorphism with domain $\sum D_{J(i)}$. Then

$g_J(\sum D_{J(i)}) \subset M_J$, $g_J(\mathrm{Bd}(D_{J(i)})) = f(\mathrm{Bd}(D_{J(i)}))$ and $\rho(f, g_J) < \epsilon$ since $M_J$ is of diameter less than $\epsilon$.

Defining $g = f$ on $S^2 - \sum\sum D_{J(i)}$ and $g = g_J$ on each set $\sum D_{J(i)}$ such that $F(\sum D_{J(i)}) \subset M_J$ completes the proof of Lemma 2.

We begin the proof of the theorem by defining $M$ as the set of components of point inverses $f^{-1}(t)$ for $t \in f(S^2)$. No element of $M$ separates $S^2$ since the nondegenerate elements of $M$ are components of $K(f)$. By Theorem 3.40 [3], there is a monotone mapping $m$ from $S^2$ onto $S^2/M$ and a light mapping $\lambda$ from $S^2/M$ onto $f(S^2)$ such that $f = \lambda m$. We show that $d(\lambda) = d(m) = 0$ which implies that $d(\lambda m) = d(f) = 0$.

By (2.1)′ (p. 171 of [4]), $S^2/M$ is topologically $S^2$.

To show that $d(\lambda) = 0$, it is sufficient to show that dimension $\lambda(K(\lambda)) \leqq 0$. For, if dimension $\lambda(K(\lambda)) = -1$, $\lambda$ is a homeomorphism and $d(\lambda) = 0$. If dimension $\lambda(K(\lambda)) = 0$, dimension $K(\lambda) = 0$ since $\lambda$ is light and $K(\lambda) \subset \lambda^{-1}\lambda(K(\lambda))$. Thus $\lambda$ satisfies the conditions of Lemma 2 and $d(\lambda) = 0$. Therefore, suppose $y \in K(\lambda)$. Then $\lambda^{-1}\lambda(y)$ and $m^{-1}\lambda^{-1}\lambda(y)$ are nondegenerate. Since $f^{-1}\lambda(y) = m^{-1}\lambda^{-1}(y)$, $f^{-1}\lambda(y)$ is nondegenerate and $\lambda(y)$ is an element of $f(K(f))$. Therefore, $\lambda(K(\lambda)) \subset f(K(f))$. Since dimension $f(K(f)) = 0$, dimension $\lambda(K(\lambda)) \leqq 0$, thus showing $d(\lambda) = 0$.

To show $d(m) = 0$, we observe that if $P$ is a component of $m(K(m))$, $m^{-1}(P)$ is connected and contained in a component of $K(m)$. But, since the image of every component of $K(m)$ is a point, $P$ is a point. Therefore, $m(K(m))$ is 0-dimensional and, since it is compact, can be covered by a finite collection of mutually disjoint disks of arbitrarily small diameters with $m(K(m))$ contained in the interiors of the disks. The inverse of each disk is a disk since it lies in $S^2$ and its boundary is a simple closed curve. Thus, the mapping $m'$, which is equal to $m$ on the complement of the disks and maps each disk homeomorphically onto its image under $m$, is a homeomorphism and $\rho(m, m')$ is arbitrarily small. Thus, $d(m) = 0$.

## Bibliography

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