GENERALIZED CONSERVATION LAWS

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1. **Introduction.** Let $(u^1, u^0, \cdots, u^n)$ form a (local) coordinate system belonging to the ring A of germs of analytic functions at some point P of an analytic manifold, and let $u^0(P)=u^1(P)= \cdots =u^n(P)=0$. Denote by $\mathcal{E}$ the localization of the module of differential forms on the manifold; $\mathcal{E}$ is in fact a free A-module with generators $(du^0, du^1, \cdots, du^n)$ where d is the operation of exterior differentiation. If $\boldsymbol{h}$ and $\boldsymbol{k}$ are endomorphisms of $\mathcal{E}$, we shall call an element $\theta \in \mathcal{E}$ a conservation law for $\boldsymbol{h}$ and $\boldsymbol{k}$ if and only if θ, $\boldsymbol{h}\theta$, and $\boldsymbol{k}\theta$ are all (locally) exact forms. This definition generalizes the idea of a conservation law for a single endomorphism $\boldsymbol{h}$. The reader should see [4] or [5] for example.

The problem considered in this paper is that of determining all conservation laws $\theta \in \mathcal{E}$ for a given $\boldsymbol{h}$ and $\boldsymbol{k}$. As in [6] it is convenient to impose some conditions on $\boldsymbol{h}$ and $\boldsymbol{k}$. Specifically we will require that $\boldsymbol{h}$ and $\boldsymbol{k}$ have distinct eigenvalues and certain (Nijenhuis) tensors derived from $\boldsymbol{h}$ and $\boldsymbol{k}$ vanish. For example, the vanishing of the tensor $[\boldsymbol{h}, \boldsymbol{h}]$ is an integrability condition which, with appropriate hypotheses, ensures the existence of a basis of conservation laws for $\mathcal{E}$.

The preceding problem is identical to the following one concerning systems of linear homogeneous partial differential equations in three independent variables. Let the matrix (h^i_j) represent $\boldsymbol{h}$ and the matrix (k^i_j) represent $\boldsymbol{k}$ with respect to the basis $(du^0, du^1, \cdots, du^n)$ of $\mathcal{E}$. If one considers the system

$$\frac{\partial u^i}{\partial t} + h^i_\alpha \frac{\partial u^\alpha}{\partial x} + k^i_\alpha \frac{\partial u^\alpha}{\partial y} = 0, \qquad i = 0, 1, \cdots, n, \tag{1.1}$$

where the use of Greek indices indicates a summation from 0 to n, and (h^i_j) and (k^i_j) depend only on $u^0, u^1, \cdots, u^n$, the problem is one of finding a suitable nonsingular matrix which multiplies the matrix equation (1.1) on the left so as to obtain a new system

$$\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} V + \frac{\partial}{\partial y} W = 0 \tag{1.2}$$

where U, V, and W are column vectors depending on $u^0, u^1, \cdots, u^n$.

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The system (1.2) is a system of conservation laws. We note that in particular the system (1.1) is said to contain a single conservation law

$$\frac{\partial f}{\partial t}+\frac{\partial g}{\partial x}+\frac{\partial h}{\partial y}=0 \tag{1.3}$$

provided that functions $v_0, v_1, \cdots, v_n$ of $u^0, u^1, \cdots, u^n$ can be found such that

$$v_\alpha \frac{\partial u^\alpha}{\partial t}=\frac{\partial f}{\partial t}, \qquad v_\alpha h^\alpha_\beta \frac{\partial u^\beta}{\partial x}=\frac{\partial g}{\partial x}$$

and

$$v_\alpha k^\alpha_\beta \frac{\partial u^\beta}{\partial y}=\frac{\partial h}{\partial y}.$$

Since many properties are known for systems of partial differential equations which contain conservation laws it is of interest to know when systems such as (1.1) contain conservation laws or can be expressed entirely in terms of conservation laws. For example, in the study of hyperbolic partial differential equations one finds that the notion of a weak solution of (1.2) depends on the fact that (1.2) is a system of conservation laws (see [2]).

2. **Preliminaries.** In this section some definitions which appear elsewhere are repeated. As in §1, let $\mathcal{E}$ be the free A-module on $(n+1)$ generators $(du^0, du^1, \cdots, du^n)$. One defines du^i by giving its value on an element $\partial/\partial u^j$ of $E=\operatorname{Hom}(\mathcal{E}, A)$. That is, $\langle \partial/\partial u^j, du^i\rangle=\delta^i_j\in A$ for $i, j\in(0, 1, \cdots, n)$. The elements of E are called vector fields even though A is a ring of germs. Since E is also a free A-module on generators $(\partial/\partial u^0, \partial/\partial u^1, \cdots, \partial/\partial u^n)$, any element $M\in E$ can be written uniquely in the form $M=m_\alpha\, \partial/\partial u^\alpha$ for functions $m_i\in A$. Vector fields are derivations of A into itself: that is, for any $a, b\in A$ and $M\in E$, $M(ab)=a(Mb)+b(Ma)$.

The elements of E will be denoted by $L, M, \cdots$, while the elements of $\mathcal{E}$ will be denoted by $\theta, \phi, \cdots$, and so forth. An endomorphism h of $\mathcal{E}$ will be distinguished from its adjoint by writing Lh when h acts on elements of E and $h\theta$ when h acts on elements of $\mathcal{E}$. That is, $\langle Lh, \theta\rangle=\langle L, h\theta\rangle$ for any $L\in E$ and $\theta\in\mathcal{E}$.

If $\Lambda^*\mathcal{E}$ denotes the exterior algebra generated by $\mathcal{E}$, we note that an element $h\in\operatorname{Hom}_A(\mathcal{E}, \mathcal{E})$ induces homomorphisms of $\Lambda^*\mathcal{E}$ which is a direct sum

$$\Lambda^*\mathcal{E} = \Lambda^0\mathcal{E}\oplus\Lambda^1\mathcal{E}\oplus\cdots\oplus\Lambda^n\mathcal{E}$$

where $\Lambda^0\mathcal{E}=A$ and $\Lambda^1\mathcal{E}=\mathcal{E}$. The induced transformations which are of primary interest to us are defined by

$$h^{(1)}(\theta \wedge \phi) = h\theta \wedge h\phi, \tag{2.1}$$

$$h^{(2)}(\theta \wedge \phi) = h\theta \wedge \phi + \theta \wedge h\phi, \tag{2.2}$$

for any $\theta, \phi \in \mathcal{E}$. The map $h^{(0)}$ is the identity on $\mathcal{E}\wedge\mathcal{E}$. The equation (2.1) defines a homomorphism of Λ^*E, and (2.2) defines a derivation. That these maps are well defined may easily be verified.

The Nijenhuis torsion $[h, h]$ of h is defined for any $\theta\in\mathcal{E}$ as an element of $\mathrm{Hom}_A(\mathcal{E}, \mathcal{E}\wedge\mathcal{E})$ by the formula

$$[h, h]\theta = -h^{(2)}d\theta + h^{(1)}dh\theta - h^{(0)}dh^2\theta. \tag{2.3}$$

Again it is easy to establish that $[h, h]$ is well defined. If $[L, M]$ denotes the Lie bracket of any two vector fields L and M, then a dual characterization of $[h, h]$ as an element of $\mathrm{Hom}_A(E\wedge E, E)$ may also be obtained by setting

$$\begin{aligned}(L \wedge M)[h, h] \\ = [L, M]h^2 + [Lh, Mh] - [Lh, M]h - [L, Mh]h.\end{aligned} \tag{2.4}$$

It should be observed that if h is singular one can always obtain a nonsingular transformation h^* such that $[h, h]=[h^*, h^*]$ simply by setting $h^*=h+\alpha I$, where α is a suitable nonzero constant.

Finally, for any endomorphisms h and k of $\mathcal{E}$ the differential concomitant $[h, k]\in\mathrm{Hom}(\mathcal{E}, \mathcal{E}\wedge\mathcal{E})$ for any $\theta\in\mathcal{E}$ is defined by the formula

$$\begin{aligned}[h, k]\theta = \tfrac{1}{2}\{ & -[h^{(1)}k^{(1)} - (hk)^{(1)}]d\theta \\ & + [h^{(1)}dk\theta + k^{(1)}dh\theta] - [dhk\theta + dkh\theta]\}.\end{aligned} \tag{2.5}$$

A dual formula which appears in [1] can also be established. It is given for any L and $M\in E$ by

$$\begin{aligned}(L \wedge M)[h, k] = \tfrac{1}{2}\{ & [L, M]hk + [L, M]kh + [Lh, Mk] \\ & + [Lk, Mh] - [Lh, M]k - [Lk, M]h \\ & - [L, Mh]k - [L, Mk]h\}.\end{aligned} \tag{2.6}$$

3. **The conservation law problem.** The result obtained in this section can be stated as follows. Let h and $k\in\mathrm{Hom}_A(\mathcal{E}, \mathcal{E})$ and the eigenvalues of h and k respectively be distinct. If $[h, h]$ and $[h, k]$ vanish and $hk=kh$, then θ is a conservation law for h if and only if θ is a conservation law for k. Since the problem of obtaining conservation laws for a single endomorphism has been treated in several papers (see

[4], [5] and [6]), one is then in a position to apply the results of these investigations and obtain conservation laws for a collection of endomorphisms $\boldsymbol{h}_1, \boldsymbol{h}_2, \cdots, \boldsymbol{h}_p$ under appropriate hypotheses.

If the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_n$ of $\boldsymbol{h}$ are distinct, then $\boldsymbol{h}$ is cyclic on $\mathcal{E}$ (its adjoint is also cyclic on E) and the assumption that $\boldsymbol{hk}=\boldsymbol{kh}$ implies $\boldsymbol{k}$ is a polynomial in $\boldsymbol{h}$. That is, $\boldsymbol{k}$ has the form

$$\text{(3.1)} \qquad \boldsymbol{k} = f(\boldsymbol{h}) = b_0 I + b_1 \boldsymbol{h} + \cdots + b_n \boldsymbol{h}^n, \qquad b_i \in A,$$

and any eigenvectors of $\boldsymbol{h}$ are also eigenvectors of $\boldsymbol{k}$ while

$$\text{(3.2)} \quad \beta_i = f(\lambda i) = b_0 + b_1\lambda_i + \cdots + b_n\lambda_i^n, \qquad i \in (0, 1, \cdots, n),$$

are the eigenvalues of $\boldsymbol{k}$.

THEOREM 3.1. *If the eigenvalues $\lambda_0, \cdots, \lambda_n$ of $\boldsymbol{h}$ are distinct, if $\boldsymbol{hk}=\boldsymbol{kh}$ and if $[\boldsymbol{h}, \boldsymbol{h}]=[\boldsymbol{h}, \boldsymbol{k}]=0$, then there exist coordinates $v^0, \cdots, v^n$ such that $\partial/\partial v^i$ $(i=0, 1, \cdots, n)$ are eigenvector fields of $\boldsymbol{h}$ and $\boldsymbol{k}$, and the corresponding eigenvalues λ_i and β_i of $\boldsymbol{h}$ and $\boldsymbol{k}$ respectively are functions of v^i alone.*

PROOF. Let L_i and L_j $(i\neq j)$ be eigenvector fields of $\boldsymbol{h}$. Since $\boldsymbol{hk}=\boldsymbol{kh}$, L_i and L_j are also eigenvector fields of $\boldsymbol{k}$. Thus if one sets $L_i\boldsymbol{h}=\lambda_iL_i$, $L_j\boldsymbol{h}=\lambda_jL_j$, $L_i\boldsymbol{k}=\beta_iL_i$, and $L_j\boldsymbol{k}=\beta_jL_j$ and applies equation (2.6) the result is

$$\begin{aligned} 2(L_i \wedge L_j)[\boldsymbol{h}, \boldsymbol{k}] = {} & [L_i, L_j]\{(\boldsymbol{h} - \lambda_i I)(\boldsymbol{k} - \beta_i I) + (\boldsymbol{h} - \lambda_j I)(\boldsymbol{k} - \beta_i I)\} \\ & + \{(\lambda_i - \lambda_j)(L_j\beta_i) + (\beta_i - \beta_j)(L_j\lambda_i)\}L_i \\ & + \{(\lambda_i - \lambda_j)(L_i\beta_j) + (\beta_i - \beta_j)(L_i\lambda_j)\}L_j. \end{aligned}$$

In the event that $\boldsymbol{h}=\boldsymbol{k}$ the above relation reduces to

$$\begin{aligned} (L_i \wedge L_j)[\boldsymbol{h}, \boldsymbol{h}] = {} & [L_i, L_j]\{(\boldsymbol{h} - \lambda_i I)(\boldsymbol{h} - \lambda_j I)\} \\ & + (\lambda_i - \lambda_j)\{(L_j\lambda_i)L_i + (L_i\lambda_j)L_j\}, \end{aligned}$$

and the existence of coordinates $v^0, \cdots, v^n$ such that $\partial/\partial v^0, \cdots, \partial/\partial v^n$ are eigenvector fields (cf. [1] and [3]) is a consequence of $[\boldsymbol{h}, \boldsymbol{h}]=0$. Moreover since $\lambda_i\neq\lambda_j$ it follows that $(L_i\lambda_j)=0$ for $i\neq j$ and consequently λ_i depends on v^i alone.

If one returns to the original formula for $2(L_i\wedge L_j)$ $[\boldsymbol{h}, \boldsymbol{k}]$ it is now clear that the vanishing of $[\boldsymbol{h}, \boldsymbol{h}]$ yields

$$2(L_i \wedge L_j)[\boldsymbol{h}, \boldsymbol{k}] = (\lambda_i - \lambda_j)\{(L_j\beta_i)L_i + (L_i\beta_j)L_j\},$$

and hence β_i does not depend on v^j if $[\boldsymbol{h}, \boldsymbol{k}]$ vanishes.

THEOREM 3.2. *Let $\boldsymbol{h}$ have distinct eigenvalues λ_j. If $\boldsymbol{hk}=\boldsymbol{kh}$ and*

$[h, h] = [h, k] = 0$ and $\theta \in \mathcal{E}$ is a conservation law for h, then it is also a conservation law for k.

PROOF. The conditions of the theorem guarantee the existence of a basis of eigenforms $(dv^0, dv^1, \cdots, dv^n)$ of $\mathcal{E}$, and any conservation law for h must have the form $\theta = \alpha_0 dv^0 + \cdots + \alpha_n dv^n$, where each function α_j depends on v^j alone. Consequently

$$k\theta = \alpha_0\beta_0 dv^0 + \alpha_1\beta_1 dv^1 + \cdots + \alpha_n\beta_n dv^n$$

and hence by Theorem 3.1 it follows that $d(k\theta) = 0$. Since closed forms are also (locally) exact, θ is then a conservation law for k.

It should be remarked that the vanishing of $[k, k]$ is a consequence of the hypotheses imposed on h and k in Theorems 3.1 and 3.2. This fact should be clear from the following argument. Since one can always obtain a generator θ of $\mathcal{E}$ which is also a conservation law (for h) simply by setting $\theta = dv^0 + dv^1 + \cdots + dv^n$ and thereby obtain a basis $(\theta, h\theta, h^2\theta, \cdots, h^n\theta)$ of $\mathcal{E}$, it suffices to compute $[k, k](h^i\theta)$ for $i = 0, 1, \cdots, n$. The vanishing of $[k, k]$ is then a consequence of the fact that $d(k^j h^i \theta) = d(\beta_\alpha^j \lambda_\alpha^i dv^\alpha) = 0$ for any pair (i, j) of nonnegative integers.

If the additional condition that k has distinct eigenvalues β_j is imposed, then one can obtain the following theorem.

THEOREM 3.3. *If h and k have distinct eigenvalues λ_j and β_j respectively, $hk = kh$, and $[h, h] = [h, k] = 0$, then any conservation law for k is also a conservation law for h.*

PROOF. Since $(dv^0, dv^1, \cdots, dv^n)$ is a basis of eigenforms for $\mathcal{E}$, any conservation law for k will have the form

$$\phi = \gamma_0 dv^0 + \gamma_1 dv^1 + \cdots + \gamma_n dv^n,$$

where the functions γ_i may be functions of more than one variable v^i. It suffices to show that $\gamma_i = \gamma_i(v^i)$, for then clearly $d(h\phi) = 0$. To simplify the notation, let us set $\partial\gamma_j/\partial v^i = \gamma_{j,i}$ and use Greek indices α and σ to indicate summations. Then the hypotheses that $d\phi = d(k\phi) = 0$ lead to the conditions

(i) $d\phi = (\gamma_{\sigma,\alpha} - \gamma_{\alpha,\sigma}) dv^\alpha \wedge dv^\sigma = 0$ and

(ii) $d(k\phi) = (\beta_\sigma \gamma_{\sigma,\alpha} - \beta_\alpha \gamma_{\alpha,\sigma}) dv^\alpha \wedge dv^\sigma = 0$

which may be rewritten as

(i′) $\gamma_{i,j} = \gamma_{j,i}$ and

(ii′) $\beta_i \gamma_{i,j} = \beta_j \gamma_{j,i}$

where $i \neq j$. Since β_j are distinct, it follows that $\gamma_{i,j} = 0$ or $\gamma_i = \gamma_i(v^i)$.

4. Conclusion. Since it is known that conservation laws exist (locally) for a single endomorphism h in the case that h has distinct eigenvalues and $[h, h]=0$, one then observes that this class of conservation laws agrees with the class of conservation laws for a given k if the conditions $[h, k]=0$, $kh=hk$, and k has distinct eigenvalues are additionally imposed. That this class can in fact be quite large may be seen from equation (2.3), which says that if θ is a conservation law for h, then $h^i\theta$ is also a conservation law for any positive integer i.

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