ON THE ANALOG OF LITTLEWOOD'S PROBLEM IN POWER SERIES FIELDS[1]

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If K is any field, we can form the field $\mathfrak{P}_K$ whose nonzero elements are expressions

$$\Xi = \sum_{i=0}^{\infty} a_{i-n}t^{n-i}$$

with $a_{i-n} \in K$, $a_{-n} \neq 0$. We let Ξ' denote the sum of those terms of Ξ for which $i > n$. Following [3], we write $|\Xi| = e^n$, $|0| = 0$ and $\|\Xi\| = |\Xi'|$. The main result of [3] is: *If K is infinite then there are θ, $\Phi \in \mathfrak{P}_K$ such that*

$$|N| \, \|N\theta\| \, \|N\Phi\| \geqq e^{-2} \tag{1}$$

for all polynomials $N \neq 0$. Baker [2] has given definite elements of $\mathfrak{P}_K$ for which the left side of (1) is bounded below, but his bound is slightly smaller than e^{-2}. We now prove an extension of the theorem of Davenport and Lewis. The present method could also be specialized to give a new proof of their result. First, we prove some lemmas about diophantine approximation in $\mathfrak{P}_K$.

Let $M = M(d; c_1, \cdots, c_m)$ be the collection of polynomials N which satisfy

$$|N| \leqq e^d, \qquad \|N\theta_i\| \leqq e^{c_i} \qquad (i = 1, \cdots, m) \tag{2}$$

where $d \geqq 0$, $c_i \leqq -1$. Let $b = \max(0, d+m+1+c_1+\cdots+c_m)$.

LEMMA 1. *M is a K-vector space of dimension at least b and at most $d+1$.*

LEMMA 2. *$M(d; \cdots) = M(d-1; \cdots) + M'$ where M' is either $\{0\}$ or a 1-dimensional space generated by a polynomial of degree d.*

PROOF. When all $c_i = -1$, (2) defines the $(d+1)$-dimensional space of all polynomials of degree at most d. Thus Lemma 1 holds in this case. If either d or one of the c_i is reduced by 1, a certain coefficient of N or $N\theta_i$ is required to be 0. This defines a linear subspace of codimension at most 1 in M. This observation both proves Lemma 2 and provides the inductive step to prove Lemma 1.

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REMARK. This proof allows us to construct spaces defined by the inequalities (2) which are 1-dimensional. When $m=1$ the generators of those spaces can be obtained by a continued fraction (cf. [**1**, §12]).

LEMMA 3. *The spaces M all have exactly the dimension b (of Lemma* 1) *if and only if*

$$|N| \cdot \prod_{i=1}^{m} \|N\theta_i\| \geqq e^{-m} \tag{3}$$

for all polynomials $N \neq 0$.

PROOF. Let $N\neq 0$ be a polynomial. The smallest M containing N (denoted $M(N)$) has b such that $e^b = |N| \cdot \Pi_{i=1}^{m} \|N\theta_i\| \cdot e^{m+1}$. Also $\dim M(N) \geqq 1$ so if it is exactly equal to b we must have (3). On the other hand, if some M has dimension greater than b, the reduction of Lemma 2 leads to a 1-dimensional M for which $b=0$. If N generates this space, it clearly can not satisfy (3).

COROLLARY. *If* (3) *holds, the M' of Lemma 2 is* 1*-dimensional whenever $b>0$.*

THEOREM: *If K is infinite, then there is a sequence $\theta_1, \theta_2, \cdots$ of elements of $\mathfrak{P}_K$ such that* (3) *holds for all polynomials $N\neq 0$ and each $m\geqq 0$. Furthermore, if $\theta_1, \cdots, \theta_m$ satisfy* (3) *there is such a sequence that begins with these terms.*

PROOF. By induction, it suffices to produce θ_{m+1} when $\theta_1, \cdots, \theta_m$ are given, since the case $m=0$ is trivially true.

By the proof of Lemma 3 and the inductive hypothesis,

$$e^{m+1} \cdot |N| \cdot \prod_{i=1}^{m+1} \|N\theta_i\| = e^b \|N\theta_{m+1}\|$$

where $b=\dim M(N)$. Thus we must show that the coefficients of $t^{-1}, \cdots, t^{-b}$ in $N\theta_{m+1}$ can not all be zero. These coefficients are linear functions on $M(N)$, so we have a linear function from $M(N)$ to K^b which we must show to be an isomorphism. If a basis is chosen for $M(N)$ this requires only that a certain determinant be nonzero.

Suppose $\theta_{m+1} = a_1 t^{-1} + a_2 t^{-2} + \cdots$. If N has degree d, the rule for multiplying elements of $\mathfrak{P}_K$ gives the coefficient of t^{-i} in $N\theta_{m+1}$ as a linear combination of $a_i, \cdots, a_{d+i}$. Furthermore, a_{d+i} must occur.

If $N_1, \cdots, N_b$ is a basis for M and γ_{ij} is the coefficient of t^{-i} in $N_j\theta_{m+1}$, we write $\Delta = \det(\gamma_{ij})$ $(1 \leqq i, j \leqq b)$. Lemma 2 and the corollary to Lemma 3 tell us that M has a basis in which $N_1, \cdots, N_{b-1}$ are a basis for $M(d-1; \cdots)$ and N_b has degree d. Thus γ_{ij} will be a linear combination of $a_1, \cdots, a_{b+d}$ with a_{b+d} occurring only in γ_{bb}. A closer look tells us that $\Delta(d; \cdots) = a_{b+d}\Delta(d-1, \cdots) +$(terms containing $a_1, \cdots, a_{b+d-1}$). There are only finitely many choices of $d, c_1, \cdots, c_m$ for which $b+d=k$. Thus if $a_1, \cdots, a_{k-1}$ have been obtained such that all Δ with $b+d<k$ are not zero, there are only a finite number of values of a_k which cause any Δ with $b+d=k$ to vanish. Since K is infinite, we can determine a θ_{m+1} so that no Δ is zero. This is all that is required to prove the theorem.

References

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