ON ISOMORPHISMS WITH SMALL BOUND

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If X and Y are locally compact Hausdorff spaces, and if we denote by $C_0(X)$ and $C_0(Y)$ the Banach spaces of continuous, complex-valued functions vanishing at infinity on X and Y respectively, then, according to the Banach-Stone Theorem, the existence of an isometry between the function spaces $C_0(X)$ and $C_0(Y)$ implies that X and Y are homeomorphic. In [1] it was shown that if we assume, in addition, that X and Y satisfy the first axiom of countability, then this theorem can be generalized by considering isomorphisms with small norm instead of isometries. The object of this paper is to show that the conclusions of [1] remain valid without the assumption of first countability.

THEOREM. *Let X and Y be locally compact Hausdorff spaces. If there exists a norm-increasing isomorphism ϕ of $C_0(X)$ onto $C_0(Y)$ with bound less than two, $\|f\| \leqq \|\phi(f)\| \leqq \|\phi\| \, \|f\|$, $f \in C_0(X)$, $\|\phi\| < 2$, then X and Y are homeomorphic.*

PROOF. We employ the notation of [1]. M will denote a real number with $\|\phi\| < 2M < 2$, and we define the constants M', N, N' by $M' = \|\phi\| - M$, $N = 1/2M$, $N' = 1 - N$. For any point $x \in X$ (resp. $y \in Y$) we denote by μ_x (resp. μ_y) the positive unit mass concentrated at the point x (resp. y). We then denote by Y_1 the set of all $y \in Y$ for which there exists an $x \in X$ with

$$\phi^* \mu_y = \alpha \mu_x + \mu, \tag{1}$$

where α is a complex number with $|\alpha| > M$, and μ is a regular Borel measure on X (element of $C_0(X)^*$) with $\mu(\{x\}) = 0$, and hence $\|\mu\| < M'$. We thus obtain a mapping ρ from Y_1 to X defined by $\rho(y) = x$, where x is associated with y by (1).

Similarly, we let X_1 represent the set of all $x \in X$ for which there exists a $y \in Y$ with

$$\phi^{*-1} \mu_x = \beta \mu_y + \mu, \tag{2}$$

where β is a complex number with $|\beta| > N$, and μ is a regular Borel measure on Y with $\mu(\{y\}) = 0$, and hence $\|\mu\| < N'$. A mapping τ of X_1 to Y is then defined by $\tau(x) = y$, where y is associated with x by (2).

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What we want to show is that $Y_1 = Y$, and that ρ is a homeomorphism of Y onto X which has τ as its inverse. In [1], this was established by a sequence of three propositions. The proofs of the first two of these depended upon an application of the dominated convergence theorem, for which some countability hypothesis was necessary. Here we establish the theorem by proving the same three propositions, but in the absence of a countability condition the method of proof is altered considerably.

PROPOSITION 1. *ρ (resp. τ) is a mapping of Y_1 (resp. X_1) onto X (resp. Y).*

PROOF. Let x be any point of X and let $\{U_i: i \in I\}$ be the family of neighborhoods of x, where the set of indices I is directed in the usual manner by set inclusion ($i_1 \leqq i_2$ if $U_{i_2} \subseteq U_{i_1}$). For each index i, let $f_{x,i}$ be an element of $C_0(X)$ with $f_{x,i}(x) = \|f_{x,i}\| = 1$, and $f_{x,i}(x') = 0$ for all $x' \in X - U_i$. Then it is clear that

$$\lim_i\, (\phi(f_{x,i}))(y) = \lim_i \int \phi(f_{x,i})d\mu_y = \lim_i \int f_{x,i}d(\phi^*\mu_y)$$

exists for all $y \in Y$, and is equal to $(\phi^*\mu_y)(\{x\})$.

We wish to show that there exists at least one $y \in Y$ such that $\lim_i |(\phi(f_{x,i}))(y)| > M$. To this end, for each $i \in I$ we will denote by S_i the subset of Y defined by

$$S_i = \{y \in Y: |(\phi(f_{x,i}))(y)| > M\}.$$

(Note that S_i is nonvoid since ϕ is norm-increasing.) We then denote by Y_x the subset of all $y \in Y$ such that there exists a net $\{y_i: i \in I\}$ in Y, with $y_i \in S_i$ for each i, which has y as a cluster point.

First of all, we claim that Y_x is a finite set. For assume that y is any element of Y_x and g_y is an element of $C_0(Y)$ with $g_y(y) = \|g_y\| = M$. Let V be the neighborhood of y defined by $V = \{y' \in Y: |g_y(y')| > \|\phi\|/2\}$. For each $i \in I$ such that $S_i \cap V$ is nonvoid, we choose $y_i \in S_i \cap V$ and define the complex number λ_i by $|\lambda_i| = 1$ and $\arg \lambda_i = \arg g_y(y_i) - \arg(\phi(f_{x,i}))(y_i)$; then we have $\|\lambda_i\phi(f_{x,i}) + g_y\| > M + \|\phi\|/2$, so that $\|\lambda_i f_{x,i} + \phi^{-1}(g_y)\| > M/\|\phi\| + \frac{1}{2} > 1$.

Now $\|\phi^{-1}(g_y)\| \leqq M < 1$, so that the maximum set of the function $|\lambda_i f_{x,i} + \phi^{-1}(g_y)|$ is contained in the neighborhood W_i of x defined by $W_i = \{x' \in X: f_{x,i}(x') \neq 0\}$. Moreover, at any point x' of this maximum set $|(\phi^{-1}(g_y))(x')|$ is bounded away from zero by the positive quantity $\epsilon = M/\|\phi\| - \frac{1}{2}$. Since $y \in Y_x$, there exists a net $\{y_i\}$ in Y, with $y_i \in S_i$ for each i, that has y as a cluster point. And for each i such that $y_i \in V$, there exists a point x_i of the corresponding set

W_i in X with $|(\phi^{-1}(g_y))(x_i)| \geqq \epsilon$. Since the W_i thus obtained constitute a neighborhood basis at x, we conclude that $|(\phi^{-1}(g_y))(x)| \geqq \epsilon$.

But this clearly implies that Y_x is finite. For given any m points $y_1, \cdots, y_m$ of Y_x, we can choose the corresponding functions g_{y_k} with disjoint supports, so that for any choice of complex numbers $\gamma_1, \cdots, \gamma_m$, with $|\gamma_k| = 1$, $k = 1, \cdots, m$, we have $\| \sum_{k=1}^m \gamma_k g_{y_k} \| = M$. But if we choose the γ_k such that the numbers $\gamma_k(\phi^{-1}(g_{y_k}))(x)$ have equal arguments, we obtain

$$\left\| \phi^{-1}\left(\sum_{k=1}^m \gamma_k g_{y_k} \right) \right\| \geqq \left| \sum_{k=1}^m \gamma_k (\phi^{-1}(g_{y_k}))(x) \right| \geqq m\epsilon.$$

Hence Y_x is finite, say $Y_x = \{y_1, y_2, \cdots, y_n\}$, and we write

$$\phi^{*-1}\mu_x = \sum_{k=1}^n \beta_k \mu_{y_k} + \mu,$$

where the β_k are complex numbers and μ is a regular Borel measure on Y with $\mu(\{y_k\}) = 0$ for $k = 1, \cdots, n$. (A very simple argument shows that Y_x is nonvoid. However, since this fact will be a consequence of what is proven below, for the moment we simply set $\phi^{*-1}\mu_x = \mu$, if Y_x is void.) We now show that there exists at least one $y_k \in Y_x$ with $\lim_i |(\phi(f_{x,i}))(y_k)| > M$. For suppose that for all $y_k \in Y_x$ we had $\lim_i |(\phi(f_{x,i}))(y_k)| \leqq M$. Then, since Y_x is finite, there would be an $i_1 \in I$ such that for all $i \geqq i_1$ and all $y_k \in Y_x$ we would have $|(\phi(f_{x,i}))(y_k)| < M + (1-M)/2$. Next, by the regularity of μ, we could find a compact set $K \subseteq Y - Y_x$ such that $|\mu|(Y-K) < (1-M)/4$. ($|\mu|$ denotes the total variation of μ.) Since K is compact and disjoint from Y_x, there exists an $i_2 \in I$ such that $i \geqq i_2$ implies $|(\phi(f_{x,i}))(y)| \leqq M$ for all $y \in K$. Hence, if we choose an $i_0 \in I$ such that $i_0 \geqq i_1$ and $i_0 \geqq i_2$, then for all $i \geqq i_0$ we would obtain (noting that $\sum |\beta_k| \leqq 1$ and $\|\mu\| \leqq 1 - \sum |\beta_k|$)

$$\begin{aligned} 1 &= \int f_{x,i} d\mu_x = \int \phi(f_{x,i}) d(\phi^{*-1}\mu_x) \\ &= \sum \beta_k \int \phi(f_{x,i}) d\mu_{y_k} + \int_K \phi(f_{x,i}) d\mu + \int_{Y-K} \phi(f_{x,i}) d\mu \\ &< \left(\sum |\beta_k| \right) [M + (1-M)/2] + M\left(1 - \sum |\beta_k| \right) \\ &\quad + 2(1-M)/4 \\ &= M + \left(1 + \sum |\beta_k| \right)(1-M)/2 \leqq 1, \end{aligned}$$

which is absurd.

Hence we conclude that there is at least one point of Y_x, which we denote simply as y, such that $\lim_i |(\phi(f_{x,i}))(y)| > M$. Writing $\phi^*\mu_y = \alpha\mu_x + \mu$, where α is complex and $\mu(\{x\}) = 0$, we have

$$\lim_i (\phi(f_{x,i}))(y) = (\phi^*\mu_y)(\{x\}) = \alpha,$$

so that $|\alpha| > M$, $y \in Y_1$, and $\rho(y) = x$.

Similarly, if y is any point of Y, a net $\{g_{y,j}: j \epsilon J\}$ of elements of $C_0(Y)$, converging pointwise to the characteristic function of $\{y\}$, is defined in a manner exactly analogous to that used in defining the $f_{x,i}$. If we then define the subset T_j of X by $T_j = \{x \in X: |(\phi^{-1}(g_{y,j}))(x)| > N\}$, and define X_y to be the set of all $x \in X$ such that there exists a net $\{x_j\}$ in X, with $x_j \in T_j$ for all j, which has x as a cluster point, an argument analogous to that given above shows that

(a) X_y is finite, and

(b) for at least one point $x \in X_y$, we have $\lim_j |(\phi^{-1}(g_{y,j}))(x)| > N$.

(In order to establish (a), the constants M and $\|\phi\|/2$ which appear in the proof above may be replaced by N and $\frac{1}{2}$, respectively. While to establish (b), the constants analogous to $(1-M)/2$ and $(1-M)/4$ may be taken to be $(1-N\|\phi\|)/2$ and $(1-N\|\phi\|)/4$, respectively.) Then for the x given by (b), we find that $\phi^{*-1}\mu_x = \beta\mu_y + \mu$, β complex, $|\beta| > N$, $\mu(\{y\}) = 0$, and the statements about X_1 and τ follow.

PROPOSITION 2. *If $y \in Y_1$ and $\rho(y) = x$, then $x \in X_1$ and $\tau(x) = y$.*

PROOF. Let y belong to Y_1 and let $\{g_{y,j}: j \in J\}$ be a net of elements of $C_0(y)$, converging pointwise to the characteristic function of $\{y\}$, and defined as in the proof of the previous proposition. Let $\rho(y) = x$ and let us assume that either x is not an element of X_1, or that x belongs to X_1 but $\tau(x) \neq y$. We must then conclude that $\lim_j |(\phi^{-1}(g_{y,j}))(x)| \leqq N$. We now note that if x' is any element of X such that $\lim_j |(\phi^{-1}(g_{y,j}))(x')| > N$, then x' belongs to X_y, a finite set. Hence if we define the number P by

$$P = \sup_{x' \in X} \lim_j |(\phi^{-1}(g_{y,j}))(x')|$$

there exists an $x_1 \in X_y$ with

$$\lim_j |(\phi^{-1}(g_{y,j}))(x_1)| = P.$$

Now by assumption $x_1 \neq x$, so that there exists a $y_1 \in Y_1$, $y_1 \neq y$, with $\rho(y_1) = x_1$. This means that $\phi^*\mu_{y_1} = \alpha\mu_{x_1} + \mu$, with $|\alpha| > M$ and $\mu(\{x_1\}) = 0$. We now write

$$\mu = \sum_{k=2}^{n} \alpha_k \mu_{x_k} + \nu,$$

where $\{x_1, x_2, \cdots, x_n\}$ is the set X_y, and $\nu(\{x_k\}) = 0$, $k = 1, 2, \cdots, n$. Note that

$$\sum_{k=2}^{n} |\alpha_k| + \|\nu\| = \|\mu\| < M'.$$

Since $M > M'$, we may choose a positive number ϵ such that $(P-\epsilon)M > (P+\epsilon)M'$. We may then find a $j_1 \in J$ such that for all $j \geqq j_1$, we have $|(\phi^{-1}(g_{y,j}))(x_1)| > P-\epsilon$ and $|(\phi^{-1}(g_{y,j}))(x_k)| < P+\epsilon$, $k=2, \cdots, n$. Next, since $|\nu|(X_y) = 0$, we can find a compact set $K \subseteq X - X_y$ such that $|\nu|(X-K) \leqq [(P+\epsilon)-N]\|\nu\|$. Because K is compact and disjoint from X_y, there exists a $j_2 \in J$ such that if $j \geqq j_2$, $|(\phi^{-1}(g_{y,j}))(x')| \leqq N$ for all $x' \in K$. We then choose $j_0 \in J$ such that $j_0 \geqq j_1$, $j_0 \geqq j_2$, and such that for all $j \geqq j_0$ the support of $g_{y,j}$ does not contain the point y_1. Hence for $j \geqq j_0$, we have

$$\begin{aligned} 0 &= \int g_{y,j} d\mu_{y_1} = \int \phi^{-1}(g_{y,j}) d(\phi^* \mu_{y_1}) \\ &= \alpha \int \phi^{-1}(g_{y,j}) d\mu_{x_1} + \sum_{k=2}^{n} \alpha_k \int \phi^{-1}(g_{y,j}) d\mu_{x_k} \\ &\quad + \int_{X-K} \phi^{-1}(g_{y,j}) d\nu + \int_K \phi^{-1}(g_{y,j}) d\nu. \end{aligned}$$

But for all $j \geqq j_0$, the modulus of the first term on the right is greater than $(P-\epsilon)M$, while the modulus of the sum of the remaining terms is less than or equal to

$$\begin{aligned} &(P+\epsilon)\left(\sum_{k=2}^{n} |\alpha_k|\right) + [(P+\epsilon) - N]\|\nu\| + N\|\nu\| \\ &= (P+\epsilon)\|\mu\| < (P+\epsilon)M', \end{aligned}$$

and this contradiction completes the proof of the proposition.

PROPOSITION 3. *$Y_1 = Y$ and ρ is a homeomorphism of Y onto X.*

The proof is the same as that given for Proposition 3 of [1].

REFERENCE

1. M. Cambern, *A generalized Banach-Stone theorem*, Proc. Amer. Math. Soc. **17** (1966), 396–400.

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