SYMMETRIC MAXIMAL IDEALS IN $M(G)$

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Let $M(G)$, where G is a locally compact abelian group, denote the convolution algebra of bounded measures on G. $M(G)$ is a Banach* algebra. If G is not discrete then there are always maximal ideals in $M(G)$ which are not symmetric [6, Theorem 2]. The maximal ideals of $M(G)$ are in one to one correspondence with generalised characters [5] and those corresponding to continuous characters are certainly symmetric. The same is true of any maximal ideal which is a limit, in the usual maximal ideal space topology, of maximal ideals determined by continuous characters. Rudin [2] has asked whether these are all the symmetric maximal ideals. We shall show that if G is not discrete then there are always further symmetric maximal ideals. This result has already been given by Simon [4] for the case $G=\boldsymbol{R}$, the additive group of real numbers.

The present results can be deduced from the results in [1] where we showed that G has a compact subgroup H such that there is a $\mu\in M(G/H)$ with the properties

P_1. For any generalised character χ on G/H there is $a\in\boldsymbol{C}$ and $\Psi\in(G/H)\hat{}$ with $\chi_\mu=a\Psi$ μ almost everywhere in G/H.

P_2. $D=\{a; a\in\boldsymbol{C}, \boldsymbol{a}\in[(G/H)\hat{}\,]^-\}$, where $\boldsymbol{a}$ is the constant function $\boldsymbol{a}(t)=a$ and $^-$ indicates $\sigma(L^\infty(\mu), L^1(\mu))$ closure, contains numbers a with $0<|a|<1$ but $\{|a|; a\in D\}$ does not contain the whole of $(0, 1)$. However as the result is of intrinsic interest we first show (Theorem 1) that it can be improved and applies with G/H replaced by G. In Theorem 2 we characterise the generalised characters corresponding to symmetric maximal ideals and the main result then follows.

THEOREM 1. *Let G be a locally compact abelian nondiscrete group. Then there is a measure $\mu\in M(G)$ satisfying* P_1 *and* P_2 *(with G/H replaced by G).*

PROOF. Using the result from [1] described above the present result follows if we can deduce the existence of a suitable μ in $M(G)$ from the existence of one in $M(G/H)$.

Let $\lambda\in M(G)$ be the Haar measure of H and for $\nu\in M(G)$ define $\nu^\#\in M(G/H)$ by

$$\int_{G/H} f d\nu^\# = \int_G f(x+H)d\nu(x)$$

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for all bounded continuous functions on G/H. It is not difficult to show that $\nu \rightarrow \nu^\#$ is an isometric isomorphism of the ideal $\lambda * M(G)$ onto $M(G/H)$. Denote the inverse isomorphism by $\sigma \rightarrow \sigma_\flat$. If $\sigma \in M(G/H)$ then any function $g \in L^1(\nu_\flat)$, constant on the cosets of H, can be considered as a function $g^\#$ on G/H, $g^\# \in L^1(\nu)$ and $g^\# \cdot \nu = (g \cdot \nu_\flat)^\#$.

Certain results concerning the convolution of measures and functions summable with respect to Haar measure can be carried over to the convolution of measures in $M(H)$ with functions summable with respect to $\nu \in \lambda * M(G)$, the proofs being similar, taking into account that if $\nu \in \lambda * M(G)$ then also $|\nu| \in \lambda * M(G)$. Accordingly we state without proof

LEMMA 1. *Let* $\nu \in \lambda * M(G)$.

(i) *If* $f \in L^1(\nu)$, $\sigma \in M(H)$ *then* $f * \sigma$ $(= \int f(s-t)d\sigma(t)) \in L^1(\nu)$ *and* $(f \cdot \nu) * \sigma = (f * \sigma) \cdot \nu$.

(ii) *If* σ_α *is a net in* $M(H)$ *such that* $\sigma_\alpha \rightarrow \sigma$ *uniformly on equicontinuous sets of functions on* H *then* $\sigma_\alpha * f \rightarrow \sigma * f$ *in* $L^1(\nu)$ *norm for each* $f \in L^1(\nu)$.

Now suppose $\mu \in M(G/H)$ satisfying P_1 and P_2. We shall show that $\mu_\flat$ is the required measure on G. For each open relatively compact neighbourhood U of 0 in H find an open relatively compact V such that $V + V \subset U$ and put $a_U = \lambda(V)^{-2} c_V * c_V$ where c_V is the characteristic function of V. Then $(c_v)\hat{\ } \in l_2(H\hat{\ })$ so that $(a_U)\hat{\ } \in l_1(H\hat{\ })$ and $a_U = \sum_{\chi \in H\hat{\ }} (\int \chi(t)^{-1} a_U(t) d\lambda(t))\chi$, the series converging uniformly in H. Also by Lemma 1

$$\lim_U a_U * (f \cdot \mu_\flat) = f \cdot \mu_\flat.$$

Thus if ϕ is a multiplicative linear functional on $M(G)$ and $f \in L^1(\mu_\flat)$ then

$$\phi(f \cdot \mu_\flat) = \lim_U \sum_{\chi \in H\hat{\ }} \left(\int \chi(t)^{-1} a_U(t) d\lambda(t) \right) \phi(\chi \cdot \lambda) \phi(f \cdot \mu_\flat).$$

Hence if $\phi(\chi \cdot \lambda) = 0$ for all $\chi \in H\hat{\ }$ then $\phi(f \cdot \mu_\flat) = 0$ for all $f \in L^1(\mu_\flat)$, $\chi_{\mu_\flat} = 0$ $\mu_\flat$ almost everywhere and P_1 with G/H replaced by G is trivially satisfied. If $\phi(\chi \cdot \lambda) \neq 0$ for some $\chi_0 \in H\hat{\ }$, since the $\chi \cdot \lambda$ $(\chi \in H\hat{\ })$ are orthogonal idempotents we have $\phi(\chi_0 \cdot \lambda) = 1$ and $\phi(\chi \cdot \lambda) = 0$ for $\chi \neq \chi_0$. Thus in this case

$$\phi(f \cdot \mu_\flat) = \phi(\chi_0 \cdot \lambda * f \cdot \mu_\flat).$$

By [3, Theorem 2.1.4], χ_0 can be extended to a continuous character χ on G. Since $\nu \rightarrow \chi \cdot \nu_\flat$ is an isomorphism of $M(G/H)$ onto

$\chi\cdot\lambda * M(G)$, by $\phi^{\#}(\nu)=\phi(\chi\cdot\nu_\flat)$ the restriction of ϕ to $\chi\cdot\lambda * M(G)$ defines a multiplicative linear functional $\phi^{\#}$ on $M(G/H)$. Hence if $f\in L^1(\mu_\flat)$ then

$$\phi(f\cdot\mu_\flat) = \phi(\chi\cdot\lambda * f\cdot\mu_\flat) = \phi^{\#}([\chi^{-1}\cdot(\chi\cdot\lambda) * f]^{\#}\cdot\mu)$$

so that if θ is the generalised character on G/H corresponding to $\phi^{\#}$ and if $\theta_\mu = a\Psi$, $a\in\mathbf{C}$, $\Psi\in(G/H)\hat{}$ then

$$\begin{aligned}\phi(f\cdot\mu_\flat) &= a\int_{G/H}[\chi^{-1}(\chi\cdot\lambda * f)]^{\#}\Psi d\mu\\ &= a\int_G\chi^{-1}(s)\int_H f(s-t)\chi(t)d\lambda(t)\Psi(s+H)d\mu_\flat(s)\\ &= a\int_H\int_G\chi^{-1}(s-t)f(s-t)\Psi(s-t+H)d\mu_\flat(s)d\lambda(t)\\ &= a\int_H\int_G\chi^{-1}(x)f(x)\Psi(x+H)d\mu_\flat(x)d\lambda(t)\\ &= a\int_G\chi^{-1}(x)\Psi(x+H)f(x)d\mu_\flat(x).\end{aligned}$$

Thus if $\tilde{\theta}$ is the generalised character corresponding to ϕ we have

$$\tilde{\theta}_{\mu_\flat}(x) = a\chi^{-1}(x)\Psi(x+H)$$

$\mu_\flat$ almost everywhere in G and $\mu_\flat$ satisfies P_1.

To show that $\mu_\flat$ satisfies P_2 we show that the set D defined from $\mu_\flat$ and G is the set D defined from μ and G/H with possibly 0 added.

If $\chi_\alpha\in(G/H)\hat{}$ and $\chi_\alpha\to\mathbf{a}$ in $\sigma(L^\infty(\mu), L^1(\mu))$ then for $f\in L^1(\mu_\flat)$, $(f * \lambda)^{\#}\in L^1(\mu)$ and

$$\begin{aligned}\int_G\chi_\alpha(x+H)f(x)d\mu_\flat(x) &= \int_{G/H}\chi_\alpha(f*\lambda)^{\#}d\mu\\ &\to a\int_{G/H}(f*\lambda)^{\#}d\mu\\ &= a\int_G fd\mu_\flat\end{aligned}$$

so that, considering the χ_α as functions on G, we have $\chi_\alpha\to\mathbf{a}$.

Conversely if $\chi_\alpha\to\mathbf{a}$ in $\sigma(L^\infty(\mu_\flat), L^1(\mu_\flat))$ then for any $f\in L^1(\mu)$

$$\int_{G/H} (\chi_\alpha * \lambda)^\# f d\mu = \int_G \chi_\alpha(x) f(x+H) d\mu_b(x)$$

$$\to a \int_G f(x+H) d\mu_b(x)$$

$$= a \int_{G/H} f d\mu.$$

Clearly $\chi_\alpha * \lambda = 0$ or χ_α. Considering an $f \in L^1(\mu)$ with $\int f d\mu \neq 0$ we see that if $a \neq 0$ there is a β such that for $\alpha > \beta$ $\chi_\alpha * \lambda = \chi_\alpha$ and for these values of α, $(\chi_\alpha * \lambda)^\# = (\chi_\alpha)^\# \in (G/H)\hat{}$ and $\int_{G/H} (\chi_\alpha)^\# f d\mu \to a \int f d\mu$ for all $f \in L^1(\mu)$.

THEOREM 2. *The generalised character χ corresponds to a symmetric maximal ideal if and only if for each $\sigma \in M(G)$ with $\sigma = \sigma^*$ we have*

$$\chi_\sigma(s) = [\chi_\sigma(-s)]^-$$

σ almost everywhere in G.

PROOF. Suppose χ corresponds to a symmetric ideal. Then if $\sigma = \sigma^*$ we have, for all $f \in L^1(\sigma)$, $(f \cdot \sigma)^* = f^* \cdot \sigma$, where $f^*(t) = [f(-t)]^-$, and so

$$\int \chi_\sigma f d\sigma = \left(\int \chi_\sigma f^* d\sigma \right)^-$$

$$= \int [\chi_\sigma(-t)]^- f(t) d\sigma(t).$$

Since this holds for all $f \in L^1(\sigma)$ we have $\chi_\sigma(s) = [\chi_\sigma(-s)]^-$ for σ almost all s in G. Conversely if $\nu \in M(G)$ and the generalised character χ satisfies $\chi_\sigma(s) = [\chi_\sigma(-s)]^-$ then put $\sigma = |\nu| + |\nu|^*$ so that $\sigma = \sigma^*$, $\nu = f \cdot \sigma$ and $\nu^* = f^* \cdot \sigma$ where $f, f^* \in L^1(\sigma)$. We then have

$$\int \chi_\sigma f d\sigma = \left[\int [\chi_\sigma(-s)]^- [f(-s)]^- d\sigma(s) \right]^-$$

$$= \left[\int \chi_\sigma f^* d\sigma \right]^-$$

that is $\int \chi_\sigma d\nu = [\int \chi_\sigma d\nu^*]^-$ so that the multiplicative linear functional and hence the ideal are symmetric.

THEOREM 3. *$M(G)$ has symmetric maximal ideals which do not lie in $(G\hat{})^-$.*

PROOF. Take μ as in Theorem 1 and let $a \in D$ with $0 < |a| < 1$. Then, since the maximal ideal space of $M(G)$ is compact we can find a generalised character χ in $(G^\wedge)^-$ with $\chi_\mu = \boldsymbol{a}$. The maximal ideal corresponding to χ is clearly symmetric so that, by Theorem 2, $\chi_\sigma(s) = [\chi_\sigma(-s)]^-$ for $\sigma = \sigma^*$. For positive real α, $|\chi|^\alpha$ is a generalised character on G corresponding to a symmetric maximal ideal and $|\chi_\mu|^\alpha = a^\alpha 1$. By a suitable choice of α we have $|a|^\alpha \notin D$ and hence a symmetric maximal ideal not in $(G^\wedge)^-$.

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