EXISTENCE THEORY FOR CERTAIN ORDINARY DIFFERENTIAL EQUATIONS WITH A MONOTONE SINGULARITY[1]

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The following result arose in the course of some work [2] on extremal solutions of uniformly elliptic partial differential equations. All variables and functions here are real valued and n is a positive integer.

THEOREM. *Suppose* $G(t, x_0, \cdots, x_{n-1})$ *is a function which is* $(n+1)$-*dimensionally continuous for* $t \geqq 0$, $-\infty < x_0 < \infty, \cdots, -\infty < x_{n-1} < \infty$, *uniformly Lipschitz in* $x_0, \cdots, x_{n-1}$ *for* $t \geqq 0$, *and which is monotone nonincreasing with respect to* x_{n-1} *for* t *in some neighborhood of* 0. *Let* $p_0, \cdots, p_{n-2}$ *be continuous functions of* t *which are bounded outside every neighborhood of* 0, *and such that* $p_i(t) = o(t^{(i-n)})$ *as* $t \to 0$, $i = 0, \cdots, n-2$. *There then exists a unique solution of the initial value problem*

$$
(1) \qquad \begin{aligned}
&f \in C^n, \qquad t \geqq 0, \\
&f^{(n)}(t) = G(t, p_0 f^{(0)}, \cdots, p_{n-2} f^{(n-2)}, t^{-1} f^{(n-1)}), \qquad t > 0, \\
&f^{(i)}(0) = 0, \qquad i = 0, \cdots, n-1.
\end{aligned}
$$

We point out that it suffices to show existence and uniqueness for (1) in some neighborhood $[0, \delta]$ of the origin, $\delta > 0$. Since G is uniformly Lipschitz in $f^{(0)}, \cdots, f^{(n-1)}$ for $t \geqq \delta$, the solution could be uniquely continued from $t = \delta$ using the usual Picard existence and uniqueness theory [1]. The proof now proceeds by way of several lemmas.

LEMMA 1. *If* f *is a solution of* (1) *then* $f_n \equiv f^{(n)}(0)$ *must be the unique fixed point of the equation*

$$
(2) \qquad f_n = G(0, \cdots, 0, f_n).
$$

PROOF. If f is C^n up to the origin then

$$
(3) \qquad \begin{aligned}
t^{-1} f^{(n-1)}(t) &= f_n + o(1), \\
p_i(t) f^{(i)}(t) &= o(1), \qquad i = 0, \cdots, n-2,
\end{aligned}
$$

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as $t \to 0$. Hence, taking limits in (1) we obtain (2). The fixed point of (2) exists and is unique since G is continuous and nonincreasing in the last argument.

LEMMA 2. *Given $\epsilon > 0$ there exists an "ϵ-approximate solution" f_ϵ satisfying*

$$\begin{aligned} & f_\epsilon \in C^n, \quad t \geqq 0, \\ (4) \quad & \left| f_\epsilon^{(n)}(t) - G(t, p_0 f_\epsilon^{(0)}, \cdots, t^{-1} f_\epsilon^{(n-1)}) \right| \leqq \epsilon, \quad t > 0, \\ & f_\epsilon^{(i)}(0) = 0, \quad i = 0, \cdots, n-1. \end{aligned}$$

PROOF. Let $f_\epsilon = t^n f_n / n!$ for $0 \leqq t \leqq \eta$. Then

$$\epsilon(t) \equiv f_\epsilon^{(n)}(t) - G(t, p_0 f_\epsilon^{(0)}, \cdots, t^{-1} f_\epsilon^{(n-1)}) \tag{5}$$

is a continuous function which $\to 0$ as $t \to 0$. Choose η sufficiently small that $|\epsilon(t)| < \epsilon$ for $0 \leqq t \leqq \eta$. Extend $\epsilon(t)$ continuously to zero in some neighborhood past $t = \eta$, always keeping $|\epsilon(t)| < \epsilon$ globally. Then, for $t > \eta$, let f_ϵ be defined to be a solution of

$$\begin{aligned} & f_\epsilon^{(n)}(t) = G(t, p_0 f^{(0)}, \cdots, t^{-1} f^{(n-1)}) + \epsilon(t), \quad t > \eta, \\ (6) \quad & f_\epsilon^{(i)}(\eta) = \eta^{(n-i)} f_n / (n-i)!, \quad i = 0, \cdots, n-1. \end{aligned}$$

Such a C^n solution exists by the usual Cauchy-Peano theory [1]. Clearly f_ϵ patches together in a C^n fashion at $t = \eta$, thus giving a solution of (4) as desired.

LEMMA 3. *Let f and g be two solutions of* (4). *Let K_i be the Lipschitz constant of G with respect to X_i, $i = 0, \cdots, n-2$. Let $\delta > 0$ be sufficiently small that $K_\delta = \sup_{t \in [0,\delta]} \sum_{i=0}^{n-2} K_i p^i(t) t^{(n-i)}/(n-i)!$ is less than* 1 *and that* $[0, \delta]$ *lies within the interval of monotonicity of G with respect to its last argument. Then*

$$\begin{aligned} (7) \quad & \left| (f-g)^{(i)}(t) \right| \leqq (2\epsilon/(1 - K_\delta)) t^{(n-i)}/(n-i)!, \\ & i = 0, \cdots, n-1, \quad 0 \leqq t \leqq \delta. \end{aligned}$$

PROOF. Let $y = f - g$. Now $|y^{(n-1)}|' = y^{(n)}$ when $y^{(n-1)} > 0$ and $|y^{(n-1)}|' = -y^{(n)}$ when $y^{(n-1)} < 0$. Thus, subtracting equations for f and g and using the monotonicity of G, one sees that either $|y^{(n-1)}| = 0$, or

$$|y^{(n-1)}|' \leqq K_0 |p_0 y^{(0)}| + \cdots K_{n-2} |p_{n-2} y^{(n-2)}| + 2\epsilon. \tag{8}$$

Let $[0, \delta]^*$ be the subset of $[0, \delta]$ where $|y^{(n-1)}| \neq 0$ and let

$$m = \sup_{\tau \in [0,\delta]^*} \left| y^{(n-1)} \right|'(\tau). \tag{9}$$

One easily shows that

$$\left| y^{(i)}(t) \right| \leqq m t^{(n-i)}/(n-i)!, \quad i = 0, \cdots, n-1, \; 0 \leqq t \leqq \delta. \tag{10}$$

Inserting this for $i=0, \cdots, n-2$ into (8) and taking supremums over $[0, \delta]^*$ one has $m \leqq 2\epsilon/(1-K_\delta)$ as desired.

PROOF OF THE THEOREM. Let $\epsilon_j \to 0$ as $j \to \infty$. Then, by the a priori bounds (7), $p_0 f_{\epsilon_j}^{(0)}, \cdots, t^{-1} f_{\epsilon_j}^{(n-1)}$ are all uniformly Cauchy convergent on $[0, \delta]$. Thus, by (4) and the continuity of G in all its variables, $f_{\epsilon_j}^{(n)}$ is also uniformly Cauchy convergent on $[0, \delta]$. The limit function is therefore a C^n solution of (1) on $[0, \delta]$ and the proof is completed.

NOTE. The local existence and uniqueness on $[0, \delta]$, δ defined as in Lemma 3, in no way utilizes the Lipschitz condition on G with respect to its last argument. This condition was required only for the global uniqueness.

REFERENCES

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