SOME PROPERTIES OF HILBERT SCALES

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1. **Hilbert scales.** Suppose E_0 is a separable complex Hilbert space with inner product $(\ ,\ )_0$. Let $A: E_0 \to E_0$ be a completely continuous linear mapping with $A > 0$. Then A can be represented in the form $Au = \sum_{j=1}^{\infty} \lambda_j (u, a_j)_0 a_j$ where $\{a_j\}$ is a complete orthonormal set in E_0, $\lambda_1 \geqq \lambda_2 \geqq \cdots > 0$, and $\lim \lambda_j = 0$, (see [1, Chapter 1]). Let $E = \bigcap_{n=0}^{\infty} A^n E_0$ and for $u, v \in E$, $\alpha \in R$ define $(u, v)_\alpha = (A^{-\alpha} u, A^{-\alpha} v)_0$. Denote by E_α, E equipped with the $\|\ \|_\alpha = (\ ,\ )_\alpha^{1/2}$ topology, and let E_α denote the completion. It follows that $E = \bigcap_\alpha E_\alpha$. The family $\alpha \to E_\alpha$ is called the *Hilbert scale* defined by A, and E is called the *center* (see [2, p. 93]). We will suppose that E carries the weakest topology in which all inclusions $E \to E_\alpha$ are continuous. It follows that E is a perfect space in the sense of [1] and $\bigcup_\alpha E_\alpha = E^*$ is its dual space.

Since $\|u\|_\alpha = \|A^\beta u\|_{\alpha+\beta}$ for all $u \in E$ and $\alpha, \beta \in R$ it follows that A^β extends to an isometry from E_α to $E_{\alpha+\beta}$, which we will also denote by A^β. Also, it follows from the definition that $\{\lambda_j^\alpha a_j\}$ is a complete orthonormal set in E_α and $(u, a_j)_\alpha \lambda_j^{2\alpha} = (u, a_j)_0$ for $u \in E_\alpha$, $\alpha \geqq 0$.

An alternative method of introducing Hilbert scales follows from the observation that if i: $E_1 \to E_0$ denotes the inclusion map, then $A = (i^* i)^{1/2}$. Hence, suppose H_0 and H_1 are complex Hilbert spaces, H_1 is dense in H_0, and the inclusion map i: $H_1 \to H_0$ is completely continuous. Then i has the form $u = i(u) = \sum_{j=1}^{\infty} \alpha_j (u, b_j)_1 a_j$ where $\lim \alpha_j = 0$, and $\{a_j\}$ and $\{b_j\}$ are bases in H_0 and H_1 respectively. It follows that $i^* i \in \mathfrak{L}(H_1, H_1)$ has the form $i^* i(u) = \sum |\alpha_i|^2 (u, b_i)_1 b_i$ which extends to H_0 by setting $i^* i(v) = \sum |\alpha_i|^2 (v, a_i)_0 a_i$. The pair $\{H_0, (i^* i)^{1/2}\}$ then defines a Hilbert scale as above.

An important class of Hilbert scales are those whose centers are nuclear spaces. It is not difficult to show that if $\alpha \to E_\alpha$ is a Hilbert scale defined by A, then the center is nuclear iff $\sum_{i=1}^{\infty} \lambda_i^\sigma < \infty$ for some $\sigma > 0$.

Two simple examples are given below, and other examples can be found in [2] and [5].

(a) Suppose E denotes the set of all C^∞ mappings u: $R \to C$ having period one, and for $u, v \in E$ define $(u, v)_0 = \int_0^1 u(t) \bar{v}(t) dt$ and $(u, v)_1 = (u, v)_0 + (Du, Dv)_0$. Let E_i denote the completion of E with respect to $(\ ,\ )_i^{1/2}$. Then the inclusion map i: $E_1 \to E_0$ is completely continuous

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and $A=(i^*i)^{1/2}$ defines a Hilbert scale with center again E. Since $A^{-2}=I-D^2$ it follows that E is nuclear.

(b) Suppose H is a separable Hilbert space, $\lambda_1 \geqq \lambda_2 \geqq \cdots > 0$, and $\lim \lambda_i = 0$. Let $H_\alpha = \{a=(a_1, a_2, \cdots): a_i \in H$ and $\sum \|a_i\|^2 \lambda_i^{-2\alpha} < \infty\}$, and set $(a, b)_\alpha = \sum (a_i, b_i)\lambda_i^{-2\alpha}$. Then $\alpha \to H_\alpha$ is a Hilbert scale whose center is nuclear iff $\sum \lambda_i^\sigma < \infty$ for some $\sigma > 0$.

2. **Smoothing operators.** Suppose $\alpha \to E_\alpha$ is a Hilbert scale defined by A, and let $u \in E_\alpha$. Then $u = \sum_{i=1}^\infty (u, \lambda_i^\alpha a_i)_\alpha \lambda_i^\alpha a_i$. Define $P_n u = \sum_{i=1}^n (u, \lambda_i^\alpha a_i)_\alpha \lambda_i^\alpha a_i$. It follows that P_n: $E^* \to E$ is well defined, $P_n^2 = P_n$, $P_n A^\alpha = A^\alpha P_n$ for all $\alpha \in R$; and for $u \in E_\alpha$, $\|P_n u - u\|_\alpha \to 0$.

When the center of the scale is nuclear a subsequence $S_k = P_{n(k)}$ can be given that is very similar to the smoothing operators used in [**4**] and [**6**]. The bounds given below for the norms of these operators are a bit weaker than those used by Nash, however they are good enough to carry out the proof of his implicit function theorem (see [**3**]).

LEMMA. *Suppose $b_1 \geqq b_2 \geqq \cdots > 0$, $\tau = \inf\{\sigma: \sum_{i=1}^\infty b_i^\sigma < \infty\}$ is finite, and $\epsilon > 0$. Let $m(r)$ be the number of b_n greater than r^{-1}. Then there is a $k_0 > 0$ such that for $t > \tau + \epsilon$.*

$$\sum_{j \leq m(k)} b_j^{-t} \leqq k^{\tau+\epsilon+t}$$

and

$$\sum_{j>m(k)} b_j^t < k^{\tau+\epsilon-t}(\tau+\epsilon)/(t-\tau-\epsilon), \qquad \textit{for } k \geqq k_0.$$

PROOF. From [**1**, p. 88] we have that $\limsup(\log m(k)/\log k) = \tau$. Hence for some $k_0 > 0$ it follows that $m(k) < k^{\tau+\epsilon}$ if $k_0 \leqq k$. Therefore $\sum_{j \leqq m(k)} b_j^{-t} \leqq k^t m(k) < k^{\tau+\epsilon+t}$.

Now, $\sum_{j>m(k)} b_j^t \leqq \sum_{r=0}^\infty (m(k+(r+1)\Delta x) - m(k+r\Delta x))/(k+r\Delta x)^t$ for $k_0 \leqq k$ and $\Delta x > 0$. Letting $\Delta x \to 0$ gives

$$\sum_{j>m(k)} b_j^t \leqq \int_k^\infty x^{-t} dm(x) = x^{-t} m(x) \Big|_k^\infty + \int_k^\infty t x^{-t-1} m(x) dx$$

$$\leqq x^{\tau+\epsilon-t} \Big|_k^\infty + \int_k^\infty t x^{\tau+\epsilon-t-1} dx = k^{\tau+\epsilon-t}(\tau+\epsilon)/(t-\tau-\epsilon).$$

THEOREM. *Suppose $\alpha \to E_\alpha$ is a Hilbert scale defined by A and $\tau = \inf\{\sigma: \sum \lambda_i^{2\sigma} < \infty\}$. Given $\epsilon > 0$ there is a $K > 0$, and a subsequence $S_n = P_{k(n)}$ such that for $\beta - \alpha > \tau + 2\epsilon$*

$$\|S_n u\|_\beta / \|u\|_\alpha \leqq n n^{(\beta-\alpha)/(\tau+\epsilon)} \qquad \textit{for all } u \in E_\alpha,$$

and

$$\|(I - S_n)v\|_\alpha / \|v\|_\beta \leqq K n n^{(\alpha-\beta)/(\tau+\epsilon)} \qquad \textit{for all } v \in E_\beta.$$

PROOF. For $u \in E_\alpha$ and $\beta > \alpha$,

$$\|P_n u\|_\beta^2 = \left\| \sum_{i=1}^n (u, a_i)_\alpha \lambda_j^{2\alpha} a_j \right\|_\beta^2 \leqq \sum_{i=1}^n |(u, a_i)_\alpha|^2 \lambda_i^{4\alpha} \|a_i\|_\beta^2$$

$$\leqq \|u\|_\alpha^2 \sum_{i=1}^n \|a_i\|_\alpha^2 \|a_i\|_\beta^2 \lambda_i^{4\alpha} = \|u\|_\alpha^2 \sum_{i=1}^n \lambda_i^{2(\alpha-\beta)}.$$

For $u \in E_\beta$ and $\beta - \alpha > \tau$ we have

$$\|(I - P_n)u\|_\alpha^2 = \left\| \sum_{k>n} (u, a_k)_\beta \lambda_k^{2\beta} a_i \right\|^2 = \sum_{k>n} |(u, a_k)_\beta|^2 \lambda_k^{4\beta} \|a_k\|_\alpha^2$$

$$\leqq \|u\|_\beta^2 \sum_{k>n} \|a_k\|_\beta^2 \lambda_k^{4\beta} \|a_k\|_\alpha^2 = \|u\|_\beta^2 \sum_{k>n} \lambda_k^{2(\beta-\alpha)}.$$

Applying the above lemma gives that for $\beta - \alpha > \tau + \epsilon$ and $k \geqq k_0$ that $\|P_{m(k)}u\|_\beta / \|u\|_\alpha \leqq k^{(\tau+\epsilon+\beta-\alpha)/2}$ for $u \in E_\alpha$, and $\|(I - P_{m(k)})v\|_\alpha / \|v\|_\beta \leqq k^{(\tau+\epsilon+\alpha-\beta)/2}((\tau+\epsilon)/(\beta-\alpha-\tau-\epsilon))^{1/2}$, $v \in E_\beta$. Letting $K = ((\tau+\epsilon)/\epsilon)^{1/2}$, $n = k^{(\tau+\epsilon)/2}$, $S_n = P_{m(n^{2/(\tau+\epsilon)})}$, and assuming $\beta - \alpha > \tau + 2\epsilon$ the result follows.

3. **Commuting operators between Hilbert scales.** Suppose $\alpha \to E_\alpha$, $\beta \to F_\beta$ are Hilbert scales defined by A and B respectively, and let $\langle\ ,\ \rangle_\beta$ denote the inner product in F_β. Let $\mathfrak{L}(E, F)$ be the continuous linear maps from E to F, and define

$$\mathfrak{L}a(E, F) = \{T \in \mathfrak{L}(E, F) : TA^\alpha = B^\alpha T \quad \text{for all } \alpha \in R\}.$$

PROPOSITION. *Suppose $T \in \mathfrak{L}(E, F)$. Then for some $r \in R$ and all $\alpha \in R$, $T: E_{,\alpha} \to F_{,\alpha-r}$ is continuous. Hence T has a unique extension to an element of $\mathfrak{L}(E_\alpha, F_{\alpha-r})$ (also denoted by T), and the norms in the respective spaces are the same.*

PROOF. Since $T \in \mathfrak{L}(E, F)$ there is an r such that T: $E_{,r} \to F_{,0}$ is continuous, and we may write $T \in \mathfrak{L}(E_r, F_0)$. For $u \in E$ and any α, $\|Tu\|_{\alpha-r} = \|B^{r-\alpha}Tu\|_0 = \|TA^{r-\alpha}u\|_0 \leqq \|T\| \, \|A^{r-\alpha}u\|_r = \|T\| \, \|u\|_\alpha$. Hence T: $E_{,\alpha} \to F_{,\alpha-r}$ is continuous and extends to an element of $\mathfrak{L}(E_\alpha, F_{\alpha-r})$ whose norm is bounded by $\|T\|$. Replacing α by $-\alpha$ then gives the equality of the norms.

For $T \in \mathfrak{L}a(E, F)$ we will say that ord $T \leqq r$ if $T \in \mathfrak{L}(E_\alpha, F_{\alpha-r})$

for all $\alpha \in R$. For subsets A, $B \subset E$ write $A \perp B$ iff $(a, b)_\alpha = 0$ for all $a \in A$, $b \in B$, and $\alpha \in R$.

PROPOSITION. *Suppose $T \in \mathfrak{L}a(E, F)$ and K denotes the kernel of T in E. Then $E = K \oplus K^\perp$.*

PROOF. By the above Proposition $T \in \mathfrak{L}(E_\alpha, F_{\alpha-r})$. Let K_α denote the kernel of T in E_α and $K_\alpha^\perp$ its orthogonal complement. The result then follows from the facts that $K = \bigcap K_\alpha$ and $K^\perp = \bigcap K^\perp$.

PROPOSITION. *Suppose $T \in \mathfrak{L}a(E, F)$ is surjective. Define $L: F \to E$ by $Lv = m$ iff $m \in K^\perp$ and $Tm = v$. Then $L \in \mathfrak{L}a(F, E)$ and $TLu = u$ for all $u \in F$. Moreover if* ord $T \leqq r$ *and* ord $L \leqq s$, *then $r + s \geqq 0$ and $TE_\alpha \supset F_{\alpha+s}$.*

PROOF. For the closed graph theorem it follows that L is continuous. Since $Lv = m$, $A^\alpha Lv = A^\alpha m$. But $Tm = v$ gives $B^\alpha v = B^\alpha Tm = TA^\alpha m$ and since $A^\alpha m \in K^\perp$, $LB^\alpha v = A^\alpha m$ so that $A^\alpha L = LB^\alpha$, and $L \in \mathfrak{L}a(F, E)$.

For the second part we may assume that T is bijective by restricting T to $K^\perp$. If $K^\perp$ is finite dimensional then ord $T =$ ord $L = 0$. If $K^\perp$ is infinite dimensional and $r + s < 0$ it follows that ord $I < 0$. This would imply that for some $\epsilon > 0$ and $K > 0$, $\|u\|_\epsilon \leqq K\|u\|_0$ for all $u \in E$, which is impossible. It then follows that $TE_\alpha \supset F_{\alpha+\beta}$.

In the case of example (a) above $\mathfrak{L}a(E, F)$ includes linear differential operators with constant coefficients, but not variable coefficients. Nonetheless since smoothing operators exist for Hilbert scales with nuclear centers a Nash type implicit function theorem can be given, and they therefore seem an appropriate setting for certain nonlinear problems.

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