CONVEXITY AND MINIMAL GERSCHGORIN SETS[1]

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1. The Gerschgorin circle theorem gives bounds for the eigenvalues of an $n \times n$ complex matrix $A = (a_{i,j})$ in terms of the diagonal elements and the moduli of the off-diagonal elements. Thus these bounds apply equally well to any matrix in the class

$$(1)\quad \Omega_A = \left\{ B = (b_{i,j}) \mid b_{i,i} = a_{i,i}, \; |b_{i,j}| = |a_{i,j}|, 1 \leqq i, j \leqq n \right\}.$$

The problem of how to determine all eigenvalues of matrices in Ω_A was solved by R. S. Varga and the author [6], [4] by the introduction of minimal Gerschgorin sets, $G^{\phi}(\Omega_A)$, to be defined in §3.

The original proofs depended strongly upon the Perron-Frobenius theory of nonnegative matrices. In this note, we give a new derivation of the main results of [4] using a lemma of V. Klee on convex sets [3].

I am indebted to D. E. Bzowy for the proof of Theorem 2.

2. In real n-dimensional space R_n we define the following subsets.

$$(2)\qquad H = \left\{ \mathbf{x} \in R_n \,\middle|\, \sum_{i=1}^{n} x_i = 1, x_i \geqq 0, 1 \leqq i \leqq n \right\},$$

$$(3)\qquad Q_j = \left\{ \mathbf{x} \in H \mid x_j = 0 \right\}, \qquad 1 \leqq j \leqq n.$$

Given n^2 constants $c_{i,j} \geqq 0$, $1 \leqq i, j \leqq n$, let

$$(4)\qquad M_{i,j} = \left\{ \mathbf{x} \in H \,\middle|\, c_{i,j} x_j \leqq \sum_{k \neq j} c_{i,k} x_k \right\}, \qquad 1 \leqq i, j \leqq n,$$

$$(5)\qquad S_j = \bigcap_{i=1}^{n} M_{i,j}, \qquad 1 \leqq j \leqq n.$$

The sets H, Q_j, S_j, $M_{i,j}$, are compact convex sets in R_n and, for all $i, j = 1, \cdots, n$,

$$(6)\qquad Q_j \subset S_j \subset M_{i,j}.$$

We give two theorems relating the sets S_j, $M_{i,j}$, and H.

THEOREM 1. *Let ϕ be any permutation of the integers $1, \cdots, n$. Then the following conditions are equivalent:*

$$(7)\qquad \bigcap_{i=1}^{n} M_{i,\phi(i)} \neq \varnothing,$$

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$$\bigcup_{i=1}^{n} M_{i,\phi(i)} = H. \tag{8}$$

PROOF. (7) implies (8): Let $y \in H$ and $z \in \bigcap_{i=1}^{n} M_{i,\phi(i)}$. We may assume that $y \neq z$, and choose k such that $\min_{z_i \neq 0} y_i/z_i = y_{\phi(k)}/z_{\phi(k)} = \lambda < 1$. Then $\tilde{y} = (1-\lambda)^{-1}(y - \lambda z)$ is in H and $\tilde{y}_{\phi(k)} = 0$. Thus $\tilde{y} \in Q_{\phi(k)} \subset M_{k,\phi(k)}$, and, since $M_{k,\phi(k)}$ is convex, $y = (1-\lambda)\tilde{y} + \lambda z \in M_{k,\phi(k)}$.

(8) implies (7): This is a consequence of the following lemma of V. Klee [**3**]: *If* $(j+1)$ *closed convex sets in* R_n *have convex union and any* j *have a common point, then there is a point common to all.* By assumption $\bigcup_{i=1}^{n} M_{i,\phi(i)} = H$ which is convex, and from (6) $\bigcap_{i\neq j} M_{i,\phi(i)} \supset \bigcap_{i \neq j} Q_{\phi(i)} \neq \emptyset$, so that any $(n-1)$ sets $M_{i,\phi(i)}$ have nonempty intersection. Thus $\bigcap_{i=1}^{n} M_{i,\phi(i)} \neq \emptyset$.

THEOREM 2. *The following three conditions are equivalent*:

(9) $H = \bigcup_{j=1}^{n} S_j$.

(10) $\bigcap_{1 \leq i,j \leq n} M_{i,j} \neq \emptyset$.

(11) *For any permutation* ϕ of $1, \cdots, n$,

$$H = \bigcup_{i=1}^{n} M_{i,\phi(i)}.$$

PROOF. (9) implies (10): The n closed convex sets S_j have convex union H. Any $(n-1)$ of the S_j have a common point, since by (6), $\emptyset \neq \bigcap_{j \neq i} Q_j \subset \bigcap_{j\neq i} S_j$. Thus by Klee's lemma, $\emptyset \neq \bigcap_{j=1}^{n} S_j = \bigcap_{1 \leq i,j \leq n} M_{i,j}$.

(10) implies (11): This follows immediately from Theorem 1.

(11) implies (9): If $H \neq \bigcup_{j=1}^{n} S_j$, then $\exists \mathbf{x} \in H$ such that $\mathbf{x} \notin S_j$, for $j = 1, \cdots, n$. Consequently, for each $j = 1, \cdots, n$, we can find $\phi(j)$ such that $\mathbf{x} \notin M_{\phi(j),j}$, or by (4) that $c_{\phi(j),j}x_j > \sum_{k \neq j} c_{\phi(j),k}x_k$. From this, it follows that, for each $j = 1, \cdots, n$ and $k \neq j$, $\mathbf{x} \in M_{\phi(j),k}$ and thus, that if $k \neq j$, $\phi(k) \neq \phi(j)$. Hence ϕ is a permutation and $\mathbf{x} \notin \bigcup_{j=1}^{n} M_{\phi(j),j}$ contradicting (11).

It can be shown that, if $H \neq \bigcup_{j=1}^{n} S_j$, the permutation ϕ constructed above such that $H \neq \bigcup_{j=1}^{n} M_{\phi(j),j}$ is unique [**5**, Theorem 4].

Theorems 1 and 2 may be generalized to apply to sets of the form

$$\overline{M}_{i,j} = \left\{ x \in H \;\middle|\; c_{i,j,j}x_j \leq \sum_{k \neq j} c_{i,j,k}x_k \right\}, \qquad 1 \leq i, j \leq n, \tag{4'}$$

where $c_{i,j,k} \geq 0$, $1 \leq i, j, k \leq n$, provided

$$c_{i,j,j}x_j > \sum_{l \neq j} c_{i,j,l}x_l \quad \text{implies} \quad c_{i,k,k}x_k \leq \sum_{l \neq k} c_{i,k,l}x_l \tag{12}$$

for any $i, j=1, \cdots, n$ and $k\neq j$. The proofs for the general case are identical.

3. We briefly recall some definitions from [4]. For any complex matrix A, let Ω_A be given by (1) and

$$S(\Omega_A) = \{z \mid \det(zI - B) = 0 \quad \text{for some } B \in \Omega_A\}. \tag{13}$$

If ϕ is any permutation of $1, \cdots, n$ and $\mathbf{x}\in H$, we define

$$G_i^{\phi}(\mathbf{x}) = \left\{\sigma \,\middle|\, |a_{i,\phi(i)}-\sigma\delta_{i,\phi(i)}|\, x_{\phi(i)} \leqq \sum_{j\neq\phi(i)} |a_{i,j}-\sigma\delta_{i,j}|\, x_j\right\}, \quad 1 \leqq i \leqq n, \tag{14}$$

$$G^{\phi}(\mathbf{x}) = \bigcup_{i=1}^{n} G_i^{\phi}(\mathbf{x}), \tag{15}$$

$$G^{\phi}(\Omega_A) = \bigcap_{\mathbf{x}\in H} G^{\phi}(\mathbf{x}). \tag{16}$$

The main results of [4] were a characterization of $G^{\phi}(\Omega_A)$ and a proof that

$$S(\Omega_A) = \bigcap_{\phi} G^{\phi}(\Omega_A). \tag{17}$$

The former is a simple consequence of Theorem 1 and we will derive the latter from Theorem 2.

THEOREM 3. *$\sigma\in G^{\phi}(\Omega_A)$ if and only if $\exists\mathbf{x}\in H$ such that, for $i=1, \cdots, n$,*

$$|a_{i,\phi(i)} - \sigma\delta_{i,\phi(i)}|\, x_{\phi(i)} \leqq \sum_{j\neq\phi(i)} |a_{i,j} - \sigma\delta_{i,j}|\, x_j. \tag{18}$$

PROOF. For any $i, j=1, \cdots, n$, let

$$M_{i,j}(\sigma) = \left\{\mathbf{x} \in H \,\middle|\, |a_{i,j} - \sigma\delta_{i,j}|\, x_j \leqq \sum_{k\neq j} |a_{i,k} - \sigma\delta_{i,k}|\, x_k\right\}. \tag{19}$$

By (14), (15), for any $\mathbf{x}\in H$, $\sigma\in G_i^{\phi}(\mathbf{x})$ if and only if $\mathbf{x}\in M_{i,\phi(i)}(\sigma)$, and $\sigma\in G^{\phi}(\mathbf{x})$ is equivalent to $\mathbf{x}\in \bigcup_{i=1}^{n} M_{i,\phi(i)}(\sigma)$. Thus, $\sigma\in G^{\phi}(\Omega_A) = \bigcap_{\mathbf{x}\in H} G^{\phi}(\mathbf{x})$ if and only if $\bigcup_{i=1}^{n} M_{i,\phi(i)}(\sigma)=H$, which is equivalent to $\bigcap_{i=1}^{n} M_{i,\phi(i)}(\sigma)\neq\varnothing$, by Theorem 1. This completes the proof of our theorem, which is equivalent to Theorem 1 of [4].

To prove (17), we first give a characterization of $S(\Omega_A)$.

LEMMA. *$\sigma\in S(\Omega_A)$ if and only if $\exists\mathbf{x}\in H$ such that, for $1\leqq i, j \leqq n$,*

$$|a_{i,j} - \sigma\delta_{i,j}|\, x_j \leqq \sum_{k\neq j} |a_{i,k} - \sigma\delta_{i,k}|\, x_k. \tag{20}$$

PROOF. $\sigma \in S(\Omega_A)$ if and only if $\exists B \in \Omega_A$ and $y \neq 0$, such that $(B-\sigma I)y=0$. Taking components and using the definition (1) of Ω_A, we obtain, for $1 \leqq j \leqq n$,

$$0 = \sum_{k=1}^{n} (b_{j,k} - \sigma\delta_{j,k})y_k = \sum_{k=1}^{n} | a_{j,k} - \sigma\delta_{j,k} | \, | y_k | \exp(i\theta_{j,k}),$$

where the $\theta_{j,k}$ are real numbers and $\theta_{j,j}=0$, $1 \leqq j, k \leqq n$. Thus for each $j=1, \cdots, n$, the n complex numbers,

$$\lambda_{j,m} = \sum_{k=1}^{m} | a_{j,k} - \sigma\delta_{j,k} | \, | y_k | \exp(i\theta_{j,k}), \qquad m = 1, \cdots, n,$$

form the vertices of a closed polygon in the complex plane with sides $\alpha_{j,k} = | a_{j,k} - \sigma\delta_{j,k} | | y_k |$. It is well known that, given any set of n nonnegative numbers $\alpha_1, \cdots, \alpha_n$, there exists a closed polygon with sides α_κ if and only if no α_k exceeds the sum of the others, $\alpha_k \leqq \sum_{j\neq k} \alpha_j$ for $k=1, \cdots, n$ [5, Lemma 5]. Thus $\sigma \in S(\Omega_A)$ if and only if $\exists y \neq 0$, such that $\alpha_{j,k} \leqq \sum_{j \neq k} \alpha_{j,i}$, for $1 \leqq j, k \leqq n$. This is equivalent to (20) with $x_j = | y_j | / \sum_{k=1}^{n} | y_k |$, $j=1, \cdots, n$.

THEOREM 4. $S(\Omega_A) = \bigcap_\phi G^\phi(\Omega_A)$.

PROOF. From the lemma and (19), $\sigma \in S(\Omega_A)$ is equivalent to $\bigcap_{1 \leq i,j \leq n} M_{i,j}(\sigma) \neq \varnothing$. By Theorem 2, this is equivalent to $H = \bigcup_{i=1}^{n} M_{i,\phi(i)}(\sigma)$, for each permutation ϕ of $1, \cdots, n$, which, in turn is equivalent to $\sigma \in G^\phi(\Omega_A)$, for each ϕ, or $\sigma \in \bigcap_\phi G^\phi(\Omega_A)$.

If $\sigma \notin S(\Omega_A)$, then there exists a permutation ψ such that $\sigma \notin G^\psi(\Omega_A)$ and $H \neq \bigcup_{i=1}^{n} M_{i,\psi(i)}(\sigma)$. By a remark following Theorem 2, ψ is the only permutation for which this can occur. Consequently, since the $G^\phi(\Omega_A)$ are closed, the boundary of $S(\Omega_A)$ is the union of the boundaries of the sets $G^\phi(\Omega_A)$, which implies Corollary 1 of [4].

These results may be extended. If, for $1 \leqq i, j \leqq n$,

$$(21) \qquad \overline{M}_{i,j}(\sigma) = \left\{ x \in H \,\middle|\, \big| |\sigma| (-1)^{\delta_{i,j}}\delta_{i,j} + a_{i,j} \big| x_j \leqq \sum_{k \neq j} \big| |\sigma| (-1)^{\delta_{i,j}}\delta_{i,k} + a_{i,k} \big| x_k \right\}$$

then condition (12) is satisfied. The generalizations of Theorems 1 and 2 then imply analogues of Theorems 3 and 4, for matrices in the set

$$(22) \qquad \overset{\circ}{\Omega}_A = \{ B = (b_{i,j}) \mid | b_{i,j} | = | a_{i,j} |, 1 \leqq i,j \leqq n \}.$$

This yields the main results of [5].

It is also possible to prove the analogue of Theorem 3, for Gerschgorin sets with partitionings [1], [2] using a similar argument. However, in this case, there is no analogue to the lemma characterizing $S(\Omega_A)$ and the extension of Theorem 4 is not generally true. The best results for this case have been given by Johnston [2].

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