ON A SEMIPRIMARY RING

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Let R be a ring with 1 having radical (Jacobson) N. R is called *semiprimary* [2, p. 56] if and only if R/N satisfies the minimum condition for right ideals. If M is a right R-module, a submodule A of M is called *small* [5] if $A+B=M$ for any submodule B of M implies $B=M$. A submodule A of M is called large [3] if $A\cap B=0$ for any submodule B of M implies $B=0$. A right ideal in R is called small or large if I is small or large as a submodule of the right regular R-module R_R. A *projective cover* [1] of M is an epimorphism of a projective module onto M such that its kernel is small. The main results of this paper are the following theorems:

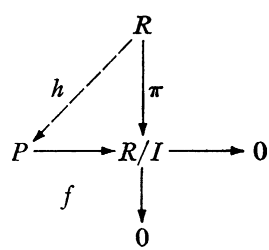
THEOREM 1. *Every irreducible (right) R-module has a projective cover if and only if R is semiprimary and for any nonzero idempotent $x+N$ in R/N there is a nonzero idempotent e in R such that $ex-e\in N$.*

Theorem 1 is related to Theorem 2.1 of [1].

THEOREM 2. *If R is commutative then every irreducible R-module has a projective cover if and only if R is semiprimary and for any nonzero idempotent $x+N$ in R/N there is an idempotent $e\in R$ such that $x-e\in N$.*

LEMMA 1. *If I is a maximal right ideal of R then the right R-module R/I has a projective cover if and only if there is a nonzero idempotent $e\in R$ such that eI is small.*

PROOF. Let f be an epimorphism from a projective module P onto R/I such that the kernel of f is small in P. Since R is projective (as R_R), there is an R-homomorphism h from R into P making



where π is the natural mapping, commutative. Since for any arbitrary $p\in P$, $f(p)=\pi(x)=fh(x)$ for some $x\in R$, $p-h(x)\in\operatorname{Ker} f$. Hence

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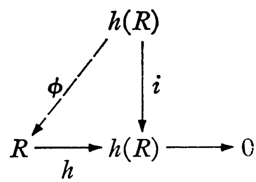
$P = \text{Ker } f + h(R)$. Since the Ker f is small, this implies that $P = h(R)$. Let $p_0 = h(1)$. Then $P = p_0 R$ and $R \xrightarrow{t_{p_0}} p_0 R \to 0$, where $t_{p_0}(x) = p_0 x$ for all $x \in R$, is direct since P is projective. Hence $\text{Ker } t_{p_0} = \{r \in R \mid p_0 r = 0\}$ is a direct summand of R. Since $p_0 = h(1)$, $\text{Ker } h = \text{Ker } t_{p_0}$. If $h(I) = 0$, then $\text{Ker } t_{p_0} = I$ and I is a direct summand of R. Hence there is a minimal right ideal J in R such that $R = J \oplus I$. Thus, by [2, p. 50], there is an idempotent $e \neq 0$ in J such that $eI = 0$ is small. If $h(I) \neq 0$ then $h(I) \subset \text{Ker } f$ since $fh(I) = \pi(I) = 0$. Thus $h(I)$ is small. Since $h(R)$ is projective, there is an R-homomorphism ϕ from $h(R)$ making



where i is the identity map, commutative. Since $h(I)$ is small, $\phi(h(I))$ is small in R by [4, p. 93]. Let $\phi(p_0) = a \in R$. Then $p_0 = h\phi(p_0) = h(a) = h(1)a = p_0 a$. Therefore, $a = \phi(p_0) = \phi(p_0 a) = a^2$ and $aI = \phi(h(I))$ is small. Clearly $a \neq 0$ since $h\phi(p_0) = p_0$. Conversely, suppose there is a nonzero idempotent e in R such that eI is small. Since $eI \subset N$ by [1, Lemma 2.4], the right ideal $(I: e) = \{r \in R \mid er \in I\}$ is I. Define a mapping g from eR onto R/I by $g(er) = r + I$ for all $er \in eR$. Since $er_1 = er_2$ then $r_1 - r_2 \in (I: e) = I$, g is well defined and clearly g is an R-homomorphism from eR onto R/I. Furthermore since eR is a direct summand of R, eR is projective and since the kernel of g is eI, which is small, g is a projective cover for R/I.

LEMMA 2. *Let I be a large maximal right ideal in R and let* $L = \{x \in R \mid xI = 0\}$. *Then* $L^2 = 0$.

PROOF. If $x \neq 0$, $y \neq 0$ are elements in L then $I \cap yR \neq 0$ and $x(yr) = 0$ for some $r \in R$ such that $yr \neq 0$ in I. If $xy \neq 0$, then $r \in I$ since the set $\{r \in R \mid (xy)r = 0\} = I$. This is impossible since $yr \neq 0$ and $y \in L$. Thus $L^2 = 0$.

PROOF OF THEOREM 1. Suppose every irreducible R-module has a projective cover. Let $\bar{I}$ be a maximal right ideal of R/N. Then there is a maximal right ideal I in R such that $\bar{I} = I/N$. By Lemma 1, there is a nonzero idempotent e in R such that eI is small. By [1, Lemma 2.4], $eI \subset N$. Since $e \notin N$, $e + N$ is a nonzero left annihilator of $\bar{I}$. Hence by Lemma 2, $\bar{I}$ cannot be large. Since $\bar{I}$ is a maximal right ideal of R/N, $\bar{I}$ must be a direct summand of R/N if $\bar{I}$ is not large. Thus by

[6, Lemma 3.1], R/N must be a semisimple ring with the minimum condition for right ideals. Now let $x\in R$ such that $x^2-x\in N$. If $x\notin N$, by Zorn's Lemma, we can construct a right ideal J^* in R with the properties that $N\subseteq J^*$, $x\notin J^*$ such that if K is a right ideal which contains J^* properly then $x\in K$. Then the right R-module $xR+J^*/J^*$ is irreducible and $(J^*: x)=\{r\in R \mid xr\in J^*\}$ is a maximal right ideal of R. Hence there is an idempotent $e\neq 0$ in R such that $e\cdot(J^*:x)\subset N$. Since $x^2-x=x(x-1)\in N$, $(x-1)\in(J^*: x)$ and $e(x-1)=ex-e\in N$. Conversely, suppose R is semiprimary and if $x+N$ is a nonzero idempotent in R/N then there is a nonzero idempotent e in R such that $ex-e\in N$. If I is a maximal right ideal of R, I/N is a maximal right ideal of R/N, and since R is semiprimary, there is a minimal right ideal K/N in R/N such that $K/N\cap I/N=N$ and $K/N\oplus I/N=R/N$ (see [4, p. 67]). Let $\bar{x}=x+N$, for some $x\in R$, be a nonzero idempotent in K/N such that $\bar{x}\cdot(I/N)=N$. By hypothesis, there is a nonzero idempotent e in R such that $ex-e\in N$. Since $xI\subset N$ and $ex-e\in N$, $eI\subset N$. Thus by Lemma 1, R/I has a projective cover.

The following corollary is related to Corollary 1 of [4, p. 76].

COROLLARY. *A ring R is local (i.e. R/N is a division ring) if and only if* 1 *is a primitive idempotent and every irreducible R-module has a projective cover.*

PROOF. If R is a local ring then 1 and 0 are only idempotents in R, and since N is the only maximal right (left) ideal in R, every irreducible R-module has a projective cover. Conversely, suppose every irreducible R-module has a projective cover and 1 is a primitive idempotent in R. By Theorem 1, R/N is a semisimple ring with the minimum condition on right ideals and if $x+N$ is a nonzero idempotent in R/N there is a nonzero idempotent e in R such that $ex-e\in N$. Since 1 is a primitive idempotent in R, $e=1$. Hence only idempotents in R/N are zero and $1+N$. Since R/N is semisimple with a minimal right ideal, this implies that R is a local ring.

PROOF OF THEOREM 2. We only need to prove that if R is commutative such that every irreducible R-module has a projective cover then idempotents modulo N can be lifted. We first prove that if $x+N$ is an idempotent such that $(x+N)(R/N)$ is a minimal ideal in R/N then $x-e\in N$ for some idempotent e in R. Let J^* be as in the proof of Theorem 1. Since $xR+N\supset J^*\supset N$ and $xR+N/N$ is a minimal ideal of R/N, $J^*=N$ since J^* is properly contained in $xR+N$. As in the case of the proof of Theorem 1, there is an idempotent e in R such that $e\cdot(J^*: x)=e\cdot(N: x)\subset N$. Now $(N: ex)=(N: x)=(N: e)$ since $(N: x)$ is a maximal ideal and $(N: ex)\supset(N: x)\supset(N: e)\supset(N: ex)$. Thus

$(1-e)\in(N\colon e)=(N\colon x)$ and $x-xe\in N$. Since $ex-e\in N$, this implies that $x-e\in N$. Now let $g=g^2$ in R such that $xg\in N$. Since $e-x\in N$, $eg\in N$. Let $e'=e-eg$. Then $g\cdot e'=0$ and $e'\cdot e'=(e-eg)(e-eg)=e-eg-eg+eg=e'$. $e'+N=e+N=x+N$. It is well known that if R/N is a semisimple ring with the minimum condition then $1+N=(x_1+N)+(x_2+N)+\cdots+(x_n+N)$ for some positive integer n where $x_i-x_i^2\in N$, $i=1, 2, \cdots, n$, $x_ix_j\in N$ if $i\neq j$ and $(N\colon x_i)$, for each i, is a maximal right ideal (see [2, p. 46 and p. 50]). By the above argument, we can choose an orthogonal set of idempotents $e_1, e_2, \cdots, e_n$ in R such that $x_i-e_i\in N$, $i=1, 2, \cdots, n$, and $1+N=(e_1+N)+(e_2+N)+\cdots+(e_n+N)$. Now let $y+N$ be an arbitrary nonzero idempotent in R/N. Then $y+N=(e_1y+N)+(e_2y+N)+\cdots+(e_ny+N)$ and $e_iy\cdot e_jy\in N$ if $i\neq j$ and $(N\colon e_iy)$ is a maximal ideal for all i such that $e_iy\notin N$. There is an orthogonal set of idempotents $a_1, a_2, \cdots, a_n$ in R such that $y-(a_1+a_2+\cdots+a_n)^2\in N$ and $(a_1+a_2+\cdots+a_n)^2=(a_1+a_2+\cdots+a_n)$.

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