ON SOME CHARACTERISTIC PROPERTIES OF SELF-INJECTIVE RINGS

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A ring with unit element is said to be *left self-injective* if and only if every (left) R-homomorphism of a left ideal of R into R can be given by the right multiplication of an element of R. In [2], Ikeda-Nakayama introduced the following conditions in a ring R with unit element:

(A) Every (left) R-homomorphism of a principal left ideal of R into R may be given by the right multiplication of an element of R.

(A_0) Every (left) R-homomorphism of a principal left ideal L of R into a residue module R/L', of R modulo a left ideal L', may be obtained by the right multiplication of an element, say c, of R: $x \to xc \pmod{L'}$, $(x \in L)$.

(B) If I is a finitely generated right ideal in R, then the set of right annihilators of the set of left annihilators of I is I.

(B*) If I is a principal right ideal in R, then the set of right annihilators of the set of left annihilators of I is I.

We introduce another condition:

(C) If F is a finitely generated left free R-module and M is a cyclic submodule of F then any R-homomorphism of M into R can be extended to a R-homomorphism of F into R.

In this paper, we shall prove the following: In a ring with 1, (B) holds if and only if (C) holds. If R is a ring with 1 such that every principal left ideal is projective, then the three conditions (A), (A_0) and (B) are equivalent. If R is a ring with 1 such that the right *singular ideal* (refer to [4] for definition) is zero, then R is a semisimple ring with minimum conditions on one-sided ideals if and only if R satisfies the maximum condition for annihilator right ideals and the condition (B). In particular, a regular ring R with 1 is a semisimple ring with minimum conditions on one-sided ideals if and only if it satisfies the maximum condition for annihilator right ideals. In a simple ring R with 1, the condition (B*) and the existence of a maximal annihilator left ideal in R are necessary and sufficient conditions for R to satisfy minimum conditions on one-sided ideals. In a ring with 1, the condition (B*) implies that the left singular ideal of R is, indeed, the Jacobson radical of R.

In the sequel, if X is a subset in R, we denote the set of left (right)

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annihilators of X in R by $l(X)(\gamma(X))$. In case $X=\{a\}$, one element set, then we let $l(X)=l(a)(\gamma(X)=\gamma(a))$.

THEOREM 1. *If R is a ring with* 1, *then the condition* (B) *holds in R if and only if the condition* (C) *holds in R.*

PROOF. Assume (B). Let

$$F = R \oplus R \oplus \cdots \oplus R \quad (n \text{ copies})$$

for some positive integer n and let M be a cyclic submodule of F. Then $M=Rm_0$ for some $m_0 \in F$ and $m_0=a_1 \dot{+} a_2 \dot{+} \cdots \dot{+} a_n$ for some $a_1, a_2, \cdots, a_n$ in R. Let h be a R-homomorphism of M into R. Then $h(m_0)=b$ for some $b \in R$. Let $L=\bigcap_{i=1}^n l(a_i)$. Then $L \subseteq l(b)$. Since $L=l(\sum_{i=1}^n a_iR)$, $\sum_{i=1}^n a_iR=r(L) \supseteq r(l(b))$ by (B). Hence $b=a_1s_1+a_2s_2+\cdots+a_ns_n$ for some $s_1, s_2, \cdots, s_n$ in R. Define $\bar{h}(r_1 \dot{+} r_2 \dot{+} \cdots \dot{+} r_n)=\sum_{i=1}^n r_is_i$ for any $r_1, r_2, \cdots, r_n$ in R. Then $\bar{h}$ is an R-homomorphism of F into R and $\bar{h}(m)=h(m)$ for all $m \in M$.

Assume (C). Let I be a finitely generated right ideal of R, say $I=\sum_{i=1}^n x_iR$. Since $I \subseteq \gamma(l(I))$ always, it suffices to prove $\gamma(l(I)) \subseteq I$. For each $x \in \gamma(l(I))$, $(\bigcap_{i=1}^n l(x_i)) \cdot x=0$. Define a (left) R-homomorphism f from R into the free left R-module

$$F = R \oplus R \oplus \cdots \oplus R \quad (n \text{ copies})$$

by $f(a)=ax_1 \dot{+} ax_2 \dot{+} \cdots \dot{+} ax_n$ for all $a \in R$. Let $M=f(R)$. Define the R-homomorphism g from M into R by $g(f(a))=ax$ for all $a \in R$. We need to show that g is indeed well defined. If $f(a)=f(b)$, $a, b \in R$, then $a-b \in \bigcap_{i=1}^n l(x_i)$ since

$$f(a) = ax_1 \dot{+} ax_2 \dot{+} \cdots \dot{+} ax_n = bx_1 \dot{+} bx_2 \dot{+} \cdots \dot{+} bx_n = f(b).$$

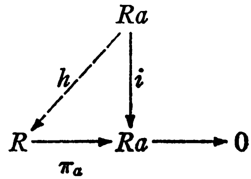
Hence $a-b \in l(x)$ and $ax=bx$. Since M is a cyclic submodule of F, by (C) we may extend g to $\bar{g}$ which is an R-homomorphism of F into R. Now

$$\begin{aligned} x = \bar{g}f(1) &= \bar{g}(x_1 \dot{+} x_2 \dot{+} \cdots \dot{+} x_n) = \bar{g}(x_1 \dot{+} 0 \dot{+} 0 \dot{+} \cdots \dot{+} 0) \\ &\quad + \bar{g}(0 \dot{+} x_2 \dot{+} 0 \cdots \dot{+} 0) + \cdots + \bar{g}(0 \dot{+} 0 \dot{+} \cdots \dot{+} 0 \dot{+} x_n) \\ &= x_1\bar{g}(1 \dot{+} 0 \dot{+} 0 \dot{+} \cdots \dot{+} 0) + x_2\bar{g}(0 \dot{+} 1 \dot{+} 0 \cdots \dot{+} 0) + \cdots \\ &\quad + x_n\bar{g}(0 \dot{+} 0 \dot{+} \cdots \dot{+} 0 \dot{+} 1). \end{aligned}$$

Thus $x \in I$.

THEOREM 2. *If R is a ring with* 1 *such that every principal left ideal is projective then the three conditions* (A), (A_0) *and* (B) *are equivalent.*

PROOF. Since (B) implies (A) always in any ring with 1 by [2, (i) of Theorem 1], we shall prove that (A)$\Rightarrow$(A_0)$\Rightarrow$(B). Let a be a nonzero element of R. Then Ra is projective. Hence the following diagram,



where i is the identity mapping, $\pi_a(x) = xa$ for all $x \in R$ and h is an R-homomorphism of Ra into R, is commutative. By (A), $h(a) = ar_0$ for some r_0 in R. Now $ar_0a = h(a)a = \pi_a h(a) = a$. Thus R is a regular ring and by [2, Theorem 3], (A)$\Rightarrow$(A_0). Since (A_0) implies that R is a regular ring by [2, Theorem 3], and any finitely generated right ideal in a regular ring is a principal right ideal generated by an idempotent element (see, for example, [6, Lemma 15, p. 710]), (A_0) implies (B).

LEMMA.[1] *Let R be a ring with unit element such that if I is a maximal right ideal in R then there is a nonzero right ideal K in R such that $I \cap K = (0)$. Then R is a semisimple ring with minimum conditions on one-sided ideals.*

PROOF. Let F be the right socle of R. If $1 \notin F$, then by Zorn's Lemma there exists a maximal right ideal, say I of R such that $I \supseteq F$. Let K be a nonzero right ideal of R such that $I \cap K = (0)$. Then K must be a minimal right ideal of R since I is a maximal right ideal. Hence K is a minimal right ideal which is not contained in F. This is impossible. Thus $1 \in F$ and $F = R$. From [1, Theorem 11, p. 61], the assertion follows.

We say a ring R satisfies the *maximum condition* for *annihilator right ideals* if every nonvacuous collection of annihilator right ideals of R contains a maximal element.

THEOREM 3. *Let R be a ring with unit element such that the right singular ideal of R is zero. Then the following two statements are equivalent:*

(a) *R is a semisimple ring with minimum conditions on one-sided ideals.*

(b) *R satisfies the maximum condition for annihilator right ideals and the condition* (B).

PROOF. Assume (a). Since any semisimple ring R with minimum

[1] The author proposed this lemma as a problem in the Canad. Math. Bull. **8** (1965).

condition on right ideals satisfies the maximum condition for right ideals (see, for example, [3, p. 64]), R satisfies the maximum condition for annihilator right ideals. By [3, Structure Theorem (3), p. 12] and [2, (iii) of Theorem 1], R satisfies the condition (B). Conversely, assume (b). Let I be a maximal right ideal of R. We shall prove that $l(I)\neq 0$. Let S be a family of annihilator left ideals in R such that $L\in S$ if and only if $L=\bigcap_{i=1}^{n} l(x_i)$ for some finite number of x_i in I. Since the maximum condition on annihilator right ideals implies the minimum condition on annihilator left ideals, we may choose a minimal member, say $L_0\in S$. Let $L_0=\bigcap_{j=1}^{k} l(x_j)$ for some $x_1, x_2, \cdots, x_j\in I$. Since $\bigcap_{j=1}^{k} l(x_j)=l(x_1R+x_2R+\cdots+x_jR)$, by (B) $\gamma(L_0)=\gamma(l(x_1R+x_2R+\cdots+x_jR))=x_1R+x_2R+\cdots+x_jR\subseteq I$. Hence $L_0\neq 0$. If $x\in I$ then $l(x)\cap L_0\subseteq L_0$. Hence $l(x)\cap L_0=L_0$ since L_0 is a minimal member of S and $l(x)\cap L_0\in S$. Thus $0\neq L_0\subseteq l(x)$ and $0\neq L_0\subseteq \bigcap_{x\in I} l(x)=l(I)$. Since the right singular ideal of R is zero and $l(I)\neq 0$, there must exist a nonzero right ideal K in R such that $I\cap K=0$. Thus, by the Lemma, (a) is true.

COROLLARY. *If R is a regular ring with 1 such that it satisfies the maximum condition for annihilator right ideals, then R is a semisimple ring with minimum conditions on one-sided ideals.*

PROOF. As noted before, any finitely generated right ideal in a regular ring is a principal right ideal generated by an idempotent element. Hence the condition (B) is satisfied. By [5, p. 1386] a regular ring has zero right singular ideal. Thus by Theorem 3, the assertion is true.

THEOREM 4. *Let R be a simple ring with unit element. Then the following two statements are equivalent:*

(a) *There exists a maximal annihilator left ideal in R and* (B*) *holds in R.*

(b) *R satisfies the minimum condition on one-sided ideals.*

PROOF. Clearly (b) implies (a). Assume (a). It is well known that a simple ring with a minimal one-sided ideal is isomorphic to a dense ring of linear transformations of finite rank of a vector space over a division ring. Hence it suffices to prove an existence of a minimal one-sided ideal in R in our case since $1\in R$. We shall show that there is a maximal right ideal I in R which has zero intersection with some nonzero right ideal K in R. Suppose that if I is a maximal right ideal in R then $I\cap K\neq 0$ for any nonzero right ideal K in R. Let $a\in R$, $a\neq 0$, such that $l(a)$ is a maximal annihilator left ideal. Then $a\cdot I\neq 0$ since a simple ring with 1 has zero (right) singular ideal. Hence

$aI \cap I \neq 0$. Let $x \in aI \cap I$ such that $x \neq 0$. Then $x = ai$ for some $i \in I$ and $l(x) = l(ai) = l(a)$ since $l(a) \subseteq l(x)$ and $l(a)$ is a maximal annihilator left ideal. Now $l(a) = l(x) = l(xR)$. Hence $a \in \gamma(l(a)) = \gamma(l(xR)) = xR \subseteq I$. Thus a is contained in the intersection of all maximal right ideals I in R. That is, the Jacobson radical of R is not zero. This is impossible.

THEOREM 5. *If R is a ring with* 1 *such that* (B*) *holds in R then the left singular ideal of R is, indeed, the Jacobson radical of R.*

PROOF. Let x be a nonzero element in the left singular ideal of R. Then $0 = l(1-x)$. Otherwise, $l(x) \cap l(1-x) \neq 0$ and if $y \in l(x) \cap l(1-x)$, $y \neq 0$, then $y = yx = 0$. Hence $R = \gamma(l(1-x)) = \gamma(l((1-x)R)) = (1-x)R$ by (B*). Thus every element of the left singular ideal of R is quasi-regular. Suppose there is an element a in the Jacobson radical of R which is not contained in the left singular ideal of R. Then there is a nonzero left ideal K in R such that $K \cap l(a) = 0$. Let $k \in K$ and $k \neq 0$. Then $l(k) = l(ka)$ since $xka = 0$ if and only if $xk = 0$ for any $x \in R$. Since $l(kR) = l(k) = l(ka) = l(kaR)$, by (B*) $kR = r(l(k)) = r(l(ka)) = kaR$. Hence $k = kar$ for some $r \in R$ and $k(1-ar) = 0$. However, ar is in the Jacobson radical of R. Hence $(1-ar)x = 1$ for some $x \in R$. This implies that $k(1-ar)x = k = 0$. This is absurd. Thus the left singular ideal of R must be the Jacobson radical of R.

COROLLARY. *If R is a ring with* 1 *such that* (A) *holds in R, then the left singular ideal of R is the Jacobson radical of R.*

PROOF. By [2, (i) of Theorem 1], (A) is equivalent to (B*). Hence from Theorem 5, the assertion follows.

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