THE RELATION BETWEEN THE SEQUENCE-TO-SEQUENCE AND THE SERIES-TO-SERIES VERSIONS OF QUASI-HAUSDORFF SUMMABILITY METHODS

B. KWEE

1. Introduction. Let (H, μ_n) be a regular Hausdorff method of summability, and let

$$t_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) s_k, \tag{1}$$

$$b_n = \sum_{k=n}^{\infty} \binom{k}{n} (\Delta^{k-n} \mu_n) a_k, \tag{2}$$

where $s_k = a_0 + a_1 + \cdots + a_k$. We shall call A the summability method given by the sequence-to-sequence transformation (1), and B the summability method given by the series-to-series transformation (2). It is proved in [2] and [3] that summabilities A and B are regular.

We shall say that the transformations (1) and (2) are equivalent if the convergence of (1) for all n implies the convergence of (2) for all n, and conversely, and in either case, the sums are related by the equation

$$t_n = b_0 + b_1 + \cdots + b_n. \tag{3}$$

(1) may be written as

$$t = H^*(\mu_{n+1}) s,$$

where s, t denote the sequences (s_k), (t_k), and $H^*(\mu_{n+1})$ the matrix $(\alpha_{n,k})$, where

$$\alpha_{n,k} = \binom{k}{n} (\Delta^{k-n} \mu_{n+1}) \qquad (k \geqq n),$$
$$= 0 \qquad (k < n).$$

We shall prove the following two theorems.

THEOREM 1. *If $t_0 = b_0$, and*

$$H^*(\mu_{n+1}) \{ H^*(n+1) s \} = H^*(n+1) \{ H^*(\mu_{n+1}) s \}, \tag{4}$$

then the transformations (1) *and* (2) *are equivalent.*

THEOREM 2. *If, for all (fixed) n,*

Received by the editors May 18, 1964 and, in revised form, November 21, 1966.

$$(5)\qquad \binom{k}{n}(\Delta^{k-n}\mu_n)s_{k-1}\to 0$$

as $k\to\infty$, *then the transformations* (1) *and* (2) *are equivalent.*

2. **Proof of Theorem 1.** Let $\bar{a}$ and $\bar{b}$ denote the sequences $\{(n+1)a_{n+1}\}$ and $\{(n+1)(t_{n+1}-t_n)\}$. Then $\bar{a}=-H^*(n+1)s$, and, by (4),

$$(6)\qquad \begin{aligned}\bar{b} &= -H^*(n+1)t = -H^*(n+1)\{H^*(\mu_{n+1})s\}\\ &= -H^*(\mu_{n+1})\{H^*(n+1)s\}\\ &= H^*(\mu_{n+1})\bar{a}.\end{aligned}$$

Hence

$$(7)\qquad (n+1)(t_{n+1}-t_n)=\sum_{k=n}^{\infty}\binom{k}{n}(\Delta^{k-n}\mu_{n+1})(k+1)a_{k+1}$$

for $n\geqq 0$. Noting that $(k+1/n+1)C_{k,n}=C_{k+1,n+1}$ and replacing $k+1$ by k and $n+1$ by n, we have

$$t_n-t_{n-1}=\sum_{k=n}^{\infty}\binom{k}{n}(\Delta^{k-n}\mu_n)a_k=b_n$$

for $n\geqq 1$, and $t_0=b_0$ by hypothesis. Thus (3) is satisfied, and the transformations (1) and (2) are equivalent.

3. **Proof of Theorem** 3. Write

$$b_{n,K}=\sum_{k=n}^{K}\binom{k}{n}(\Delta^{k-n}\mu_n)a_k,$$

$$t_{n,K}=\sum_{k=n}^{K}\binom{k}{n}(\Delta^{k-n}\mu_{n+1})s_k$$

(both of these may be taken as 0 for $n>K$). If (5) holds, then, for any fixed n, we have, as $K\to\infty$

$$(8)\qquad \begin{aligned}b_{n,K} &= \sum_{k=n}^{K}\binom{k}{n}(\Delta^{k-n}\mu_n)(s_k-s_{k-1})\\ &= \sum_{k=n-1}^{K}s_k\Delta\left\{\binom{k}{n}(\Delta^{k-n}\mu_n)\right\}+o(1),\end{aligned}$$

where the Δ outside the curly bracket is taken as operating on the variable k, and the curly bracket is taken as 0 when $k=n-1$. Now using

$$\Delta^{k-n}\mu_n = \Delta^{k+1-n}\mu_n + \Delta^{k-n}\mu_{n+1}$$

we have

$$\Delta\left\{\binom{k}{n}(\Delta^{k-n}\mu_n)\right\} = \binom{k}{n}[\Delta^{k+1-n}\mu_n + \Delta^{k-n}\mu_{n+1}] - \binom{k+1}{n}\Delta^{k+1-n}\mu_n$$

(9)

$$= -\binom{k}{n-1}\Delta^{k-(n-1)}\mu_n + \binom{k}{n}\Delta^{k-n}\mu_{n+1},$$

where we take the second term on the right of (9) as meaning 0 in the case $k=n-1$, and the first as meaning 0 when $n=0$.

We deduce at once from (8) and (9) that, for fixed n,

$$t_{n,K} = b_{0,K} + b_{1,K} + \cdots + b_{n,K} + o(1)$$

as $K\to\infty$, and this proves the theorem.

4. **Examples.** Now let us apply these ideas to some examples. We shall use the following lemma which is a paraphrase of Theorem 26 in [1].

LEMMA. *If, for any sequence (p_k) which is monotonic decreasing for large enough k, $\sum_{k=n}^{\infty} a_k p_k$ exists, then*

$$\lim_{k\to\infty} p_k \sum_{l=n}^{k} a_l = 0.$$

(i) If $\mu_n = \lambda^n$ $(0<\lambda<1)$, then (1) becomes

$$t_n = \lambda^{n+1}\sum_{k=n}^{\infty}\binom{k}{n}(1-\lambda)^{k-n}s_k. \tag{10}$$

This is the circle method of summation introduced by Hardy and Littlewood. (2) becomes

$$b_n = \lambda^n \sum_{k=n}^{\infty}\binom{k}{n}(1-\lambda)^{k-n}a_k, \tag{11}$$

and (5) becomes

$$\binom{k}{n}(1-\lambda)^{k-n}s_{k-1}\to 0. \tag{12}$$

The convergence of (10) for a given n implies (12) for that n. Also, by the lemma quoted above with $p_k = C_{k,n}(1-\lambda)^{k-n}$, the convergence of (11) for a given n implies (12).

Since summability A asserts more than the convergence of (10) for all n, and summability B asserts more than the convergence of (11) for each n, we see at once that, in this case, summabilities A and B are equivalent.

(ii) If

$$\mu_n = \binom{n+r}{r}^{-1},$$

then (1) becomes

$$(13)\qquad \begin{aligned} t_n &= r(n+1)\sum_{k=n}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{(k+r+1)(k+r)\cdots(k+r-n)} s_k \\ &= \frac{n+1}{r+1}\Delta^{-r}\left\{\frac{s_n}{\binom{n+r+1}{n}}\right\}. \end{aligned}$$

This is the quasi-Cesáro transformation (C^*, r) introduced by Kuttner [4]. (2) becomes

$$(14)\qquad \begin{aligned} b_n &= r\sum_{k=n}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{(k+r)(k+r-1)\cdots(k+r-n)} a_k \\ &= \Delta^{-r}\left\{\frac{a_n}{\binom{n+r}{n}}\right\}. \end{aligned}$$

For any given n, the assertion that the series defining t_n converges is easily seen to be equivalent to

$$(15)\qquad \sum_{k=1}^{\infty} \frac{s_k}{k^2}$$

converges, while the assertion that the series defining b_n converges is equivalent to

$$(16)\qquad \sum_{k=1}^{\infty} \frac{a_k}{k}$$

converges. Condition (5) is easily seen to reduce, in the special case considered, to

$$(17)\qquad s_k = o(k).$$

By the lemma quoted above with $p_k = 1/k$, (16) implies (17). Hence, whatever r, $B \Rightarrow A$. On the other hand, it is clearly false that (15) implies (17). But summability A asserts more than the convergence of (15), since (15) merely gives the existence of t_n. Thus this does not exclude the possibility that summability A might imply (17).

What we do, in fact, have is that $A \Rightarrow B$ is true when, $r \leqq 1$, but not when $r > 1$. For recall that A is (C^*, r). It follows from the results of a paper by Kuttner [4] that $(C^*, r) \Rightarrow (C, 1)$ when $r \leqq 1$; and it is well known that $(C, 1)$ implies (17). On the other hand, if $r > 1$, let $1 \leqq \beta < \alpha < r$, and $s_k = (-1)^k (k+1)^\beta$. Then (s_k) is summable (C, α), and hence summable (C^*, r) [4]. But (17) is false. Indeed, (16) does not converge, so that b_n is not defined.

References

1. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949.

2. M. S. Ramanujan, *On Hausdorff and quasi-Hausdorff methods of summability*, Quart. J. of Math. Oxford Ser. (2) **8** (1957), 197–213.

3. ———, *Series-to-series quasi-Hausdorff transformation*, J. Indian Math. Soc. **17** (1953), 47–53.

4. B. Kuttner, *On quasi-Cesàro summability*, J. Indian Math. Soc. **24** (1960), 319–341.

University of Malaya, Kuala Lumpur, Malaysia