A THEOREM ON INTERMEDIATE REDUCIBILITIES

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Let α, β be two sets of natural numbers. Then [2] the *least upper bound* of (the Turing degrees of) α and β is the (Turing degree of the) set $J(\alpha, \beta) = \{2x \mid x \in \alpha\} \cup \{2x+1 \mid x \in \beta\}$. In general, we shall denote by $|\alpha|_T$ the Turing degree of a set α of natural numbers, and by $|\alpha|_M$ and $|\alpha|_{tt}$ the many-one and truth-table degrees, respectively, of α [5]. It is a trivial fact that $|J(\alpha, \beta)|_M$ and $|J(\alpha, \beta)|_{tt}$ are least upper bounds for the pairs $|\alpha|_M$, $|\beta|_M$ and $|\alpha|_{tt}$, $|\beta|_{tt}$, respectively. We denote by $\leqq_T$, $\leqq_M$, and $\leqq_{tt}$ the partial order of degrees in the Turing, many-one, and truth-table semilattices, respectively.

The following fact about the semilattice of Turing degrees is well known and easy to prove:

PROPOSITION. *If α, β are number sets which can be separated by disjoint recursively enumerable sets γ, δ (thus, $\alpha \subseteq \gamma$, $\beta \subseteq \delta$, and $\gamma \cap \delta = \varnothing$), then $|\alpha \cup \beta|_T = |J(\alpha, \beta)|_T$.*

The object of this note is to exhibit a pair of Theorems (and a Corollary) showing how this Proposition breaks down if we consider finer semilattices than that of the Turing degrees; specifically, if we consider the $\leqq_M$- and $\leqq_{tt}$-semilattices.

We acknowledge that our discovery of Theorem 1 was the result of brooding over Lachlan's proof, in [3], that $\leqq_{tt}$ differs, on the r.e. sets, from the $\leqq_w$ relation of Friedberg and Rogers [1]. Indeed, Lachlan's result is a corollary to Theorem 1, since, as C. G. Jockusch has pointed out to us, $J(\alpha, \beta) \leqq_w \alpha \cup \beta$ whenever α, β are disjoint r.e. sets.

THEOREM 1. *There exist disjoint r.e. sets α and β such that*

$$\alpha \cup \beta \not\leqq_M J(\alpha, \beta) \tag{1}$$

and

$$\alpha \not\leqq_{tt} \alpha \cup \beta \qquad (\textit{whence } J(\alpha, \beta) \not\leqq_{tt} \alpha \cup \beta). \tag{2}$$

PROOF. We employ a priority construction of the elementary variety (finitely many injuries per requirement). Four types of markers are used: Λ_n, Σ_n, $+$ and $*$. A number n shall be *free* at a given point in the construction just in case neither n nor any larger number bears

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or has previously borne any type of marker, up to the point in question. "α^s", "β^s" denote, respectively, the portions of α, β defined by the end of stage s; "c^s" denotes the characteristic function of $\alpha^s \cup \beta^s$. Let T_0, T_1, T_2, $\cdots$ be a recursive enumeration of all truth-table conditions. If c is the characteristic function of a set and $\langle x_1, \cdots, x_j \rangle$ is the tuple of test numbers associated with condition T_k, we write "$T_k(c(x_1), \cdots, c(x_j)) = 0$" to mean that $c(x_1), \cdots, c(x_j)$ is a minus row of the table involved in condition T_k, and "$T_k(c(x_1), \cdots, c(x_j)) = 1$" to mean that $c(x_1), \cdots, c(x_j)$ is a plus row of that table. The construction proceeds as follows:

Stage 0. Set $\alpha^0 = \beta^0 = \varnothing$ and go to Stage 1.

Stage s, $s > 0$.

Case I. $(s)_0 = 2k$. If $s = 2^{2k}$, put marker Λ_k on the smallest free number. Let r be the number bearing Λ_k. If r also bears a *, set $\alpha^s = \alpha^{s-1}$, $\beta^s = \beta^{s-1}$, and go on to Stage $s+1$. Otherwise, proceed s steps in the computation of $\phi_k(r)$, where $\{\phi_i | i = 0, 1, 2, \cdots\}$ is the standard normal-form enumeration of the 1-place partial recursive functions. If no value for $\phi_k(r)$ is obtained, set $\alpha^s = \alpha^{s-1}$, $\beta^s = \beta^{s-1}$, and go on to Stage $s+1$. Otherwise, consider whether $\phi_k(r) \in J(\alpha^{s-1}, \beta^{s-1})$. If so, set $\alpha^s = \alpha^{s-1}$, $\beta^s = \beta^{s-1}$, place a* on r, and proceed to Stage $s+1$. If $\phi_k(r) \notin J(\alpha^{s-1}, \beta^{s-1})$, two cases arise.

Case A. $\phi_k(r) = 2r$. Then set $\alpha^s = \alpha^{s-1}$, $\beta^s = \beta^{s-1} \cup \{r\}$, place a * on r, and a + on $\phi_k(r)$ if the latter is free, move all attached markers Λ_j and Σ_m with $j > k$ and $m \geqq k$ down (without disturbing their order relative to one another) from their current positions to the first available free numbers, and go to Stage $s+1$.

Case B. $\phi_k(r) \neq 2r$. Then set $\alpha^s = \alpha^{s-1} \cup \{r\}$, $\beta^s = \beta^{s-1}$, place a * on r, and a + on $\phi_k(r)$ if the latter is free, move markers as in Case A, and go to Stage $s+1$. This completes the description of Case I.

Case II. $(s)_0 = 2k+1$. If $s = 2^{2k+1}$, attach Σ_k to the smallest free number. Let r be the position of Σ_k. If r also bears a *, set $\alpha^s = \alpha^{s-1}$, $\beta^s = \beta^{s-1}$, and go to Stage $s+1$. Otherwise, compute s steps in the search for a value for $\phi_k(r)$. If no value is obtained, set $\alpha^s = \alpha^{s-1}$, $\beta^s = \beta^{s-1}$, and go to Stage $s+1$. Suppose, on the other hand, that we find $\phi_k(r) = w$. Let $\langle x_1, \cdots, x_p \rangle$ be the tuple of numbers involved in *tt*-condition T_w. If any of $x_1, \cdots, x_p$ are free, put a + on the largest such. Set c^s = the characteristic function of $\alpha^{s-1} \cup \beta^{s-1} \cup \{r\}$. If $T_w(c^s(x_1), \cdots, c^s(x_p)) = 0$, set $\alpha^s = \alpha^{s-1} \cup \{r\}$, $\beta^s = \beta^{s-1}$. If $T_w(c^s(x_1), \cdots, c^s(x_p)) = 1$, set $\alpha^s = \alpha^{s-1}$ and $\beta^s = \beta^{s-1} \cup \{r\}$. In either case, place a * on r and move all attached markers Λ_j and Σ_j with $j > k$ down (without disturbing their order relative to one another) from their current positions to the first available free numbers, and go to Stage $s+1$.

This completes the description of Case II and of Stage s ($s>0$).

We of course set $\alpha=\bigcup_n \alpha^n$, $\beta=\bigcup_n\beta^n$; it is obvious that α and β are disjoint r.e. sets. The proof of the theorem is completed by the following three lemmas, whose proofs are routine on the basis of the construction given above:

LEMMA 1. $(\forall k)$*(both* Λ_k *and* Σ_k *achieve final positions).*

LEMMA 2. $(\forall e)$*(*ϕ_e *is not a many-one reduction of* $\alpha\cup\beta$ *to* $J(\alpha, \beta)$*).*

LEMMA 3. $(\forall e)$*(*ϕ_e *is not a tt reduction of* α *to* $\alpha\cup\beta$*).*

COROLLARY. *There exist disjoint r.e. sets* α, β *such that* $\alpha\cup\beta \leqq_M J(\alpha, \beta)$ & $J(\alpha, \beta)\nleqq_{tt}\alpha\cup\beta$.

PROOF. Let α, β be as in the theorem; then $\alpha\nleqq_{tt}\alpha\cup\beta$. Let R be an infinite recursive subset of β, and let f be a 1-1 recursive function with range R. Let $\beta^*=(\beta-R)\cup f(\alpha\cup\beta)$. We claim that α, β^* have the two properties required in the statement of the corollary. First, it is clear from the definition of β^* that $\alpha\cup\beta^*\leqq_M\beta^*$; hence, we have $\alpha\cup\beta^*\leqq_M J(\alpha, \beta)$. Next, it is easy to check that $\alpha\cup\beta^*\leqq_{tt}\alpha\cup\beta$; this prevents α—and so also $J(\alpha, \beta^*)$—from being *tt*-reducible to $\alpha\cup\beta^*$.

THEOREM 2. *There exist disjoint r.e. sets* α *and* β *such that* $J(\alpha, \beta) \leqq_M\alpha\cup\beta$ *but* $\alpha\cup\beta\nleqq_M J(\alpha, \beta)$.

PROOF. We could deduce Theorem 2 immediately from a result of P. R. Young [**6**], according to which there exist disjoint, *recursively isomorphic*, noncreative recursively enumerable sets whose union is creative. However, the proof of Young's theorem is fairly involved, and we prefer a more elementary line of argument. We shall need a simple proposition whose proof appears in [**4**]: a creative set α can be extended to a creative set β in such a way that $\beta-\alpha$ is infinite and devoid of infinite recursively enumerable subsets (i.e., α is *simple in* β), whereas, if α, β, γ are any three recursively enumerable sets such that $\beta\cap\gamma=\varnothing$, $\alpha=\beta\cup\gamma$, [δ recursively enumerable & $\delta\cap\beta=\varnothing$] $\Rightarrow\delta-\alpha$ is recursively enumerable, and [δ recursively enumerable & $\delta\cap\gamma=\varnothing$]$\Rightarrow\delta-\alpha$ is recursively enumerable, then neither β nor γ can be extended to a recursively enumerable superset in such a way that the relative difference is infinite but lacks infinite recursive subsets. Moreover, it is known (as a result of close examination by C. E. M. Yates of the Friedberg decomposition procedure for r.e.

sets) that *any* nonrecursive, recursively enumerable α can be taken as the α for such a triple α, β, γ. So let α be creative, and let α, β, γ be such a triple. Then, by the above-cited proposition, neither β nor γ is creative; hence, since the creative sets constitute the maximal many-one degree for recursively enumerable sets, each of β, γ is of many-one degree strictly less than α. We claim that $\alpha \nleqq_M J(\beta, \gamma)$ holds as well. For if $\alpha \leqq_M J(\beta, \gamma)$, then $J(\beta, \gamma)$ is creative. Hence there is another creative set δ such that $J(\beta, \gamma)$ is simple in δ. Hence either $\{2x \mid x \in \beta\}$ is simple in $\delta \cap$the even numbers, or else $\{2x+1 \mid x \in \gamma\}$ is simple in $\delta \cap$the odd numbers. The first alternative implies that β is simple in a recursively enumerable set, and the second that γ is simple in a recursively enumerable set; hence neither can obtain, and our claim is proven.

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