LIFTING MODULAR REPRESENTATIONS OF FINITE GROUPS

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Let (π) be the maximal ideal of the ring R of P-integral elements of an algebraic number field K, where P is a prime of K dividing the rational prime p. The natural homomorphism from R to $\overline{K}=R/(\pi)$ induces a map $S\rightarrow\overline{S}$ from the set of representations by matrices with coefficients in R of a finite group G into the set of representations of G in $\overline{K}$. The lifting problem in modular representation theory is to determine whether for a given representation T of G in $\overline{K}$ there exists a representation S of G by R-matrices such that $\overline{S}=T$. In this paper we introduce a notion of lifting projective modular representations from characteristic p to characteristic zero and show how this concept may be applied to the lifting problem.

NOTATION. Throughout this paper G denotes a finite group of order $|G|$ and K denotes an algebraic number field which is a splitting field for G. Let p be a rational prime and let R be the ring of P-integral elements of K, where P is a prime of K dividing p. Let (π) be the maximal ideal of R and set $\overline{K}=R/(\pi)$. $\overline{K}$ is a finite field of characteristic p which is a splitting field for G. For $a\in R$, set $\bar{a}=a+(\pi)\in\overline{K}$. If $A=(a_{ij})$ is a matrix with entries in R (R-matrix) we denote by $\overline{A}$ the matrix $(\bar{a}_{ij})$. By a linear representation of G in a field L we shall understand a homomorphism from G into $GL(m, L)$ for some m. By a projective integral representation (resp. projective modular representation) of G in R (resp. $\overline{K}$) we mean a map T of G into $GL(m, K)$ (resp. $GL(m, \overline{K})$) satisfying $T(1)=1_m$, $T(g)T(h)=\alpha(g, h)\cdot T(gh)$ where $\alpha(g, h)\in R$ (resp. $\overline{K}$) and $T(g)$ has entries in R (resp. $\overline{K}$) for all $g, h\in G$. α is called the factor set associated with T. If $\alpha(g, h)=\beta(g, h)=1$ for all $g, h\in G$, T is a linear integral representation (resp. linear modular representation). We identify linear representations with projective representations having trivial factor sets. We refer the reader to [3] and [7] for the relevant theory.

DEFINITION. Let T be a projective modular representation of G in $\overline{K}$ and let α be the associated factor set. T is *projectively liftable* if there exists a projective integral representation S of G in R with factor set β such that $\overline{S}(g)=T(g)$ for all $g\in G$. If $\alpha(g, h)=\beta(g, h)=1$ for all $g, h\in G$ (i.e. S and T are linear representations), we say that T is *liftable*.

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We emphasize that, according to the above definition, if we speak of a modular representation T of G in $\overline{K}$ being liftable to an integral representation S of G in R, then both T and S must be linear representations of G.

LEMMA 1. *Let T and W be projectively equivalent projective modular representations of G of degree m in $\overline{K}$. If T is projectively liftable, then so is W.*

PROOF. By assumption there exists a function γ from G to $\overline{K}$ and a matrix $U \in GL(m, \overline{K})$ such that $U^{-1}T(g)U = W(g)\gamma(g)$ for all $g \in G$. Let $V \in GL(m, K)$ having entries in R such that $\overline{V} = U$ and let α be a function from G to R such that $\bar{\alpha}^{-1}(g) = \gamma(g)$. det V is a unit in R so V^{-1} has entries in R. Hence if S is a projective integral representation which projectively lifts T, $V^{-1}SV\alpha$ is a projective lifting of W.

The next lemma will permit us to take finite extensions of K.

LEMMA 2. *Let K_1 be a finite extension of K and let R_1 be the ring of P_1-integral elements of K_1, where P_1 is a prime of K_1 dividing the prime P of K, i.e. $P_1 \cap K = P$. Let (π_1) be the maximal ideal of R_1 and view $\overline{K}$ as a subfield of $\overline{K}_1 = R_1/(\pi_1)$. Let T be an irreducible linear modular representation of G in $\overline{K}$. If T is liftable when viewed as a $\overline{K}_1$-representation, then T is liftable as a $\overline{K}$-representation.*

PROOF. The lemma is a consequence of the fact that the decomposition matrix of G for the prime p does not depend on K [**3**, Chapter 12].

THEOREM 1. *Let G be a finite group and suppose that $p \nmid |H^2(G, E^*)|$ where E^* is the multiplicative group of an algebraic closure E of K and where G acts trivially on E^*. Let T be an irreducible linear modular representation of G in $\overline{K}$ which is projectively liftable. Then T is liftable.*

PROOF. By assumption there is a projective integral representation S of G in R with factor set α such that $\overline{S}(g) = T(g)$ for all $g \in G$ and $\bar{\alpha}(g, h) = 1$ for all $g, h \in G$. Let e be the order of α in $H^2(G, E^*)$. Then α is equivalent to α' where $\alpha'(g, h)$ is an eth root of unity for all $g, h \in G$ [**3**, p. 360]. There exists a function ρ from G to E^* such that $\alpha'(g, h) = \alpha(g, h)\rho(g\,h)\rho^{-1}(g)\rho^{-1}(h)$ for all $g, h \in G$. In view of Lemma 2 we may assume that $\rho(g) \in K$ for all $g \in G$. Let $\rho(g) = \pi^{\nu(g)}\gamma(g)$ where $\gamma(g)$ is a unit in R and $\nu(g)$ is an integer. Since $\bar{\alpha}'(g, h) \neq 0$, $\bar{\alpha}(g, h) \neq 0$ for all $g, h \in G$, $\nu(g) + \nu(h) = \nu(gh)$. Therefore $\alpha'(g, h) = \alpha(g, h)\gamma(gh)\gamma^{-1}(g)\gamma^{-1}(h)$ for all $g, h \in G$. Let $\gamma^{-1}(g) = \lambda(g)$ and set $Z(g) = \lambda(g)S(g)$ for all $g \in G$. Z is a projective integral representation of G in R with factor set α'. $\overline{Z}$ is projectively equivalent over $\overline{K}$ to T. We may assume that K contains a primitive $(|\overline{K}| - 1)$-th root of unity δ over the rationals. Define a function μ from G to the integers by $\bar{\lambda}(g)=\bar{\delta}^{\mu(g)}$ where $1\leqq\mu(g)\leqq|K|-1$. Set $\eta(g)=\delta^{-\mu(g)}$ and let $V(g)=\eta(g)Z(g)$. V is a projective integral representation of G in R such that $\overline{V}=T$. The factor set β associated with V satisfies

$$\beta(g, h) = \alpha'(g, h)\eta^{-1}(gh)\eta(g)\eta(h)$$

for all $g, h\in G$. Since $\beta(g, h)$ is a root of unity with $\bar{\beta}(g, h)=1$ for all $g, h\in G$, we see that $\{\beta\}$ has order p^b in $H^2(G, E^*)$. Since $p\nmid|H^2(G, E^*)|$ by assumption, β is equivalent to the unit factor set. Therefore there is a function τ from G to E^* such that

$$\beta(g, h) = \tau(gh)\tau^{-1}(g)\tau^{-1}(h)$$

for all $g, h\in G$. As before we may assume that $\tau(g)\in R$ for all $g\in G$. Let $W(g)=\tau(g)V(g)$. W is a projective integral representation of G in R and $\overline{W}(g)=\bar{\tau}(g)\overline{V}(g)=\bar{\tau}(g)T(g)$ for all $g\in G$. Since $\bar{\beta}(g, h)=1$ for all $g, h\in G$, $\overline{W}$ is a linear modular representation of G in $\overline{K}$. Let $\bar{\tau}(g)=\bar{\delta}^{\theta(g)}$ where $1\leqq\theta(g)\leqq|K|-1$. Let $M(g)=\delta^{\theta(g)}W(g)$ for all $g\in G$. M is a linear integral representation of G in R and $\overline{M}=T$ and so T is liftable.

We refer the reader to [**3**, p. 361] for the definition and construction of a representation-group of a finite group with respect to an algebraically closed field (see also [**7**]). A representation-group G^* of G with respect to E is a central extension of G with kernel $A\cong H^2(G, E^*)$ with the following property: Let T be a projective representation of degree m of G in E. Then if $\{u_g: g\in G\}$ is a set of coset representatives of A in G^*, there exists a projective representation T' of G in E which is projectively equivalent to T and a linear representation S of G^* with $S(a)\in E^*\cdot 1_m$ for all $a\in A$ and $S(u_g)=T'(g)$ for all $g\in G$. If S and T' have this relationship we say that S linearizes T'. Let $\overline{E}$ be an algebraic closure of $\overline{K}$. If $p\nmid|H^2(G, E^*)|$, then a representation-group for G with respect to E is also one with respect to $\overline{E}$ [**1**, Satz 2].

Definition. We say that G has *property* (p, m) if every irreducible linear modular representation of degree m of G in $\overline{K}$ is liftable.

In view of Lemma 2 we see that property (p, m) does not depend on the splitting field chosen.

Lemma 3. *Let G^* be a representation-group for G with respect to E and suppose that K is a splitting field for both G and G^*. Assume also that $p\nmid|H^2(G, E^*)|$ and G^* has property (p, m) (with respect to $\overline{K}$). Let T be an absolutely irreducible projective modular representation of degree m of G in $\overline{K}$. Then there exists a finite extension K_1 of K such that, in the context of Lemma 2, T is projectively liftable (to an R_1-representation) when viewed as a representation of G in $\overline{K}_1$.*

PROOF. As noted above G^* is a representation-group for G with respect to both E and $\overline{E}$. T is projectively equivalent over $\overline{E}$ to an irreducible projective representation T' of G in $\overline{E}$, where T' is linearizable. Let S be a linear representation of G^* in $\overline{E}$ which linearizes T'. Hence there is a finite extension L of $\overline{K}$ such that the entries of $S(h)$ and $T'(g)$ lie in L for all $h \in G^*$ and all $g \in G$ and T' is projectively equivalent to T over L. In view of Lemma 1 we may assume that $T = T'$. There exists a finite extension K_1 of K containing a valuation ring R_1 with maximal ideal (π_1) such that $L = \overline{K}_1 = R_1/(\pi_1)$. Since K_1 is a splitting field for G^*, G^* has property (p, m) with respect to $\overline{K}_1$. S is an irreducible linear modular representation of degree m of G in $\overline{K}_1$ and so S is liftable. Let V be a linear integral representation of G^* in R_1 with $\overline{V}(h) = S(h)$ for all $h \in G^*$. G^* is a central extension of G with kernel A, say. If $a \in A$, then $S(a) = \lambda(a) 1_m$ with $\lambda(a) \in \overline{K}_1$. $V(a) = \mu(a) 1_m$, with $\bar{\mu}(a) = \lambda(a)$. Let $\{u_g : g \in G\}$ be a set of coset representatives of A in G^*. Set $U(g) = V(u_g)$ for all $g \in G$. Then U is a projective lifting of T, T being viewed as a $\overline{K}_1$-representation. This proves the result.

THEOREM 2. *Let G be a finite group possessing an abelian normal subgroup A such that every proper subgroup of $B = G/A$ is p-solvable. Suppose also that $p \nmid |H^2(G, E^*)|$, $p \nmid |H^2(B, E^*)|$ and that the algebraic number field K is now a splitting field for all subquotients of G^* and B^*, where G^* and B^* are some two representation-groups of G and B respectively. If B^* possesses property (p, m) (with respect to $\overline{K}$) then so does G.*

PROOF. Let T be an irreducible linear modular representation of degree m of G in $\overline{K}$ and let T_A denote the restriction of T to A. By Clifford's Theorem we have

$$T_A \sim e(T_1 + \cdots + T_r)$$

where $T_1, \cdots, T_r$ are inequivalent conjugate irreducible linear representations of A in $\overline{K}$. Since $\overline{K}$ is also a splitting field for A and A is abelian, each of the T_i's has degree one.

Case 1. $r > 1$. Let I be the inertia group of T_1 in G and view eT_1 as a representation W of I. $I \supset A$ and I is a proper subgroup of G so I is p-solvable. W is an irreducible linear modular representation of I in $\overline{K}$ such that $W^G \sim T$, where W^G is the induced representation of W to G [**3**, p. 348]. Since I is p-solvable, there exists an irreducible linear integral representation S of I in R with $\overline{S}(g) = W(g)$ for all $g \in I$ [**6**, Theorem 6]. Let $V = S^G$. $\overline{V}$ and $\overline{S}^G = W^G \sim T$ have the same character and the same degree. Since T is irreducible, $\overline{V} \sim T$ and so T is liftable.

Case 2. $r=1$. We have $T\sim C\times D$ (Kronecker product) where C, D are irreducible projective representations of G in $\bar{E}$ such that C is one-dimensional and $D(g)=1_m$ for $g\in A$ [**3**, Theorem 51.7]. Extending K if necessary, we may assume that the entries of $C(g)$, $D(g)$ lie in $\bar{K}$ for all $g\in G$, that T is equivalent to $C\times D$ over $\bar{K}$, and that D is projectively liftable with respect to $\bar{K}$. We refer to Lemmas 2 and 3 to justify this step. C is projectively liftable since C is a 1-dimensional representation. Hence $C\times D$ is projectively liftable and it follows from Theorem 1 that T is liftable.

REMARK. The hypothesis that $p\nmid |H^2(G, E^*)|$ is satisfied, for example, when G has a cyclic Sylow p-subgroup [**5**, p. 49]. If p is a prime, $p\geqq 5$, then $LF(2, p)$ is a minimal simple group [**4**, Chapter 12]. The hypotheses of Theorem 2 are satisfied when $p\nmid |A|$, $B=G/A \cong LF(2, p)$ where $p\geqq 5$, $\bar{K}$ has characteristic p, and $m=(p-1)/2$ or $(p+1)/2$ [**2**, p. 590].

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