# A CAMERON-MARTIN TRANSLATION THEOREM FOR A GAUSSIAN MEASURE ON $C(Y)$

J. D. KUELBS[1]

1. **Introduction.** Let $Y$ be the product space $\prod_{k=1}^{\infty} [a_k, b_k]$ where $-\infty < a_k < b_k < \infty$ for $k=1, 2, \cdots$ and $b_k - a_k = O(2^{-k})$. The topology on $Y$ is the product topology and $C(Y)$ denotes the space of real valued continuous functions on $Y$ with the uniform topology. Then a Gaussian measure $m$ can be defined on $C(Y)$ which is a generalization of Wiener measure and an analogue and generalization of the Cameron-Martin translation theorem [**1**], [**2**], [**3**] for Wiener integrals is obtainable in this setting. In another paper this measure and translation theorem will be used to obtain a representation for additive functionals on $C(X)$ when $X$ is a compact metric space.

2. **The definition of the measure $m$ on $C(Y)$ and other preliminaries.** If $p_1, \cdots, p_k$ are points in $Y$ such that $p_i = (c_{i1}, c_{i2}, \cdots)$ for $i=1, \cdots, k$ then $F^{p(1,\cdots,k)}$ will denote the $k$-variate Gaussian distribution with mean zero and covariance matrix $V=(v_{ij})$ where $2v_{ij} = \prod_{n=1}^{\infty}[1+(c_{in}-a_n)\wedge(c_{jn}-a_n)]$ and by $a\wedge b$ we mean $\min(a, b)$. That $V$ is actually a covariance matrix follows immediately from the fact that the matrix $D_m = (d_{ijm})$ with

$$d_{ijm} = 1 + (c_{im} - a_m) \wedge (c_{jm} - a_m)$$

is nonnegative definite for $m=1, 2, \cdots$ and that $V$ is the Hadamard product of the $D_m$ [**4**].

If $p_1 = (c_1, c_2, \cdots)$ and $p_2 = (d_1, d_2, \cdots)$ are in $Y$ and the Gaussian random variable with distribution $F^{p(i)}$ is denoted by $Z_{p(i)}$ for $i=1, 2$ then $E[Z_{p(1)} - Z_{p(2)}]^2 \leqq C\sum_{k=1}^{\infty} |d_k - c_k|$ if $\sum_{k=1}^{\infty}(b_k - a_k)C \geqq \exp\{2\sum_{k=1}^{\infty}(b_k - a_k)\}$. Therefore, using the final remark and Theorem 1 of [**5**], we see that these conditions on the family $F^{p(1,\cdots,k)}$, $p_i \in Y$, are sufficient to assure the existence of a measure $\mu$ on the Borel sets $\mathcal{C}$ of $C(Y)$ whose finite-dimensional distributions consist of the family postulated. The triple $(C(Y), \mathcal{Q}, m)$ will denote the measure space obtained from the standard completion of $(C(Y), \mathcal{C}, \mu)$.

By an $n$-dimensional rectangular subset $\mathcal{P}$ of $Y$ we mean a finite set of points of $Y$ which is formed by taking the product of finite subsets of the first $n$ coordinates of $Y$ with the remaining coordinates all being $\alpha_n = (a_{n+1}, a_{n+2}, \cdots)$. That is, if $a_k \leqq x_1 < \cdots < x_{M_k} \leqq b_k$

Received by the editors October 15, 1966.

[1] Supported by NSF Grant GP-3483.

for $k=1, \cdots, n$ then the rectangular subset formed is $\mathcal{P}=\{p\in Y: p=(x_{i_1}, \cdots, x_{i_n}, \alpha_n), i_k=1, \cdots, M_k\}$.

If $\mathcal{P}$ is an $n$-dimensional rectangular subset of $Y$ as given above then we define

$$(2.1) \qquad Q(\mathcal{P}, u) = \sum_{i_1=1}^{M_1} \cdots \sum_{i_n=1}^{M_n} \frac{[\Delta_1 \cdots \Delta_n u(i_1, \cdots, i_n)]^2}{\Delta_{i_1}x^1 \cdots \Delta_{i_n}x^n}$$

where

$$(2.2) \text{ (a) } \Delta_j \overset{k}{x} = \overset{k}{x_j} - \overset{k}{x_{j-1}} \quad \text{with } \overset{k}{x_0} = a_k - 1$$

for $k=1, \cdots, n$ and $j=1, \cdots, M_k$,

$$(2.2) \text{ (b) } \Delta_1 \cdots \Delta_k u(i_1, \cdots, i_n) = \Delta_1 \cdots \Delta_{k-1} u(i_1, \cdots, i_k, \cdots, i_n)$$
$$-\Delta_1 \cdots \Delta_{k-1} u(i_1, \cdots, i_k - 1, \cdots, i_n) \text{ for } k = 2, \cdots, n,$$
$$\Delta_1 u(i_1, \cdots, i_n) = u(i_1, \cdots, i_n) - u(i_1 - 1, \cdots, i_n),$$

and we assume $u(i_1, \cdots, i_n)=0$ if some $i_j=0$.

Let $W(\mathcal{P}, u)=C(\mathcal{P})^{-1/2} \exp\{-Q(\mathcal{P}, u)\}$ where

$$C(\mathcal{P}) = [\Delta_1 x^1 \cdots \Delta_{M_1} x^1]^{L_1} \cdots [\Delta_1 x^n \cdots \Delta_{M_n} x^n]^{L_n} \pi^L, \qquad L = \prod_{j=1}^{n} M_j,$$

and $L_k=L/M_k$.

From these definitions several things follow. First we notice that the variable $u(i_1, \cdots, i_n)$ is associated with the point $(x_{i_1}, \cdots, x_{i_n}, \alpha_n)$ of $\mathcal{P}$ and that for any $\mathcal{P}$ the function $W(\mathcal{P}, u)$ is a Gaussian density of $L$ variables. Secondly, it can be shown that if $F^{\mathcal{P}}$ denotes the Gaussian distribution function corresponding to the density $W(\mathcal{P}, u)$ then $F^{\mathcal{P}}$ is identical to $F^{p(1,\cdots,L)}$ where $p_1, \cdots, p_L$ are the points in $\mathcal{P}$. Furthermore, if $Z(i_1, \cdots, i_n)$ denotes the Gaussian random variable associated with the point $(x_{i_1}, \cdots, x_{i_n}, \alpha_n)$ and $\Delta_1 \cdots \Delta_n Z(i_1, \cdots, i_n)$ is defined as was $\Delta_1 \cdots \Delta_n u(i_1, \cdots, i_n)$ in (2.2) then the $\Delta_1 \cdots \Delta_n Z(i_1, \cdots, i_n)$ are independent Gaussian random variables with expectation zero and variance $\frac{1}{2}\Delta_{i_1}x^1 \cdots \Delta_{i_n}x^n$ for $i_k=1, \cdots, M_k$; $k=1, \cdots, n$.

3. **A statement of the main result.** Let $Y_n=\{\prod_{k=1}^{n} [a_k, b_k]\}\times\alpha_n$ for $n=1, 2, \cdots$ and by $S_n$ denote the $2^n$ subsets of $Y_n$ formed by selecting $n-k$ of the first $n$ coordinates and setting each such $x_j$ equal to $a_j$ while the remaining $k$ coordinates among the first $n$ are allowed to vary as they do in $Y_n$. The symbol $S$ denotes $\bigcup_{n=1}^{\infty} S_n$. If $I\in S$ and $I$ has $k>0$ coordinates which vary then $\mu_I$ denotes Lebesgue measure

on $I$ when $I$ is considered as $k$ dimensional. If $I$ is the single point $\alpha_0=(a_1, a_2, \cdots)$ then $\mu_I$ is the measure obtained by placing mass one at this point.

The Borel sets of $Y$ are denoted by $\mathcal{B}$ and if $B\in\mathcal{B}$ we define $\nu(B) = \sum_{I\in S}\mu_I(B\cap I)$. Then $\nu$ is sigma-additive on $\mathcal{B}$, $\nu(Y)=\sum_{I\in S}\mu_I(I) = \prod_{k=1}^{\infty}(1+b_k-a_k)$, and $C(Y)$ is dense in $\mathcal{L}_2(Y)$ with respect to mean square convergence. In fact, polynomials in finitely many coordinates of $Y$ with rational coefficients are dense in $\mathcal{L}_2(Y)$ and, as a result, a countable orthonormal basis of polynomials exists for $\mathcal{L}_2(Y)$.

If $g$ is in $\mathcal{L}_2(Y)$ and $f_0(p)=\int_{Y(p)}g\,d\nu$ where $Y(p)=\prod_{k=1}^{\infty}[a_k, x_k]$ for $p=(x_1, x_2, \cdots)$ in $Y$ then $f_0\in C(Y)$. By $\int_Y g\,d\tilde{f}$, when $g$ is $\mathcal{L}_2(Y)$, we will mean a generalization of the Paley, Wiener, and Zygmund integral as given in [6]. The precise definition of $\int_Y g\,d\tilde{f}$ occurs later. We now state our main result. It is easy to see how it generalizes the results of [1] and [3].

THEOREM. *Let $F$ be a measurable functional. If $g$ is in $\mathcal{L}_2(Y)$ and $f_0(p)=\int_{Y(p)}g\,d\nu$ for all $p$ in $Y$ then*

$$E(F(f)) = \exp\left\{-\int_Y g^2 d\nu\right\} E\left\{F(f+f_0)\exp\left\{-2\int_Y g\,d\tilde{f}\right\}\right\} \tag{3.1}$$

*in the sense that if either integral exists both exist and they are equal. The symbol $E(\cdot)$ denotes integration with respect to the measure $m$.*

4. The following lemmas are needed in the proof of our theorem. The definitions and results involving Riemann-Stieltjes integration of functions of $n$-variables can be found in [3, pp. 412–415].

If $I\in S$ and $I$ has $n$ coordinates which vary then $\tau(I)=\bigcup_{k=0}^{n}\tau_k(I)$ where $\tau_k(I)$ is the collection of $k$-dimensional subsets of $I$ formed by choosing $n-k$ of the $n$ coordinates of $I$ which vary and setting each $x_j$ of this collection equal to $a_j$ or $b_j$ while each $x_j$ of the remaining $k$ coordinates which can vary satisfies $a_j\leqq x_j\leqq b_j$.

If $f$ is defined on $Y$ and $J\in\tau(I)$ for some $I\in S$ then $f$ is of bounded variation on $J$ if $f^J$ is of bounded variation there when $J$ is considered as being finite dimensional. Here by $f^J$ we mean $f$ restricted to $J$. The Riemann-Stieltjes integral of $g$ with respect to $f$ over $J$, $\int_J g\,df$, when $f$ and $g$ have domain $Y$ is simply the Riemann-Stieltjes integral of $g^J$ with respect to $f^J$ over $J$. When $J$ is a single point $\alpha$ it is understood that $\int_J g\,df=g(\alpha)f(\alpha)$.

By $\mathcal{P}_N$ we will denote the $n$-dimensional rectangular subset formed by the sets $L_k=\{x_j=a_k+(b_k-a_k)(j-1)/N\}$ where $j=1, \cdots, N+1$ and $k=1, \cdots, n$. The point $(x_{i_1}, \cdots, x_{i_n}, \alpha_n)\in\mathcal{P}_N$ will be denoted by $(i_1, \cdots, i_n)$ and if $f \in C(Y)$ we define $\Delta_1 \cdots \Delta_n f(i_1, \cdots, i_n)$ as in (2.2) with $f(i_1, \cdots, i_n)$ replacing $u(i_1, \cdots, i_n)$. Finally, if $f_0, f \in C(Y)$ we define

$$\begin{aligned} &H(\mathcal{P}_N, f_0, f) \\ (4.1)\qquad &= \sum_{(i_1,\cdots,i_n)\in\mathcal{P}_N} \frac{[\Delta_1 \cdots \Delta_n f_0(i_1, \cdots, i_n)\Delta_1 \cdots \Delta_n f(i_1, \cdots, i_n)]}{\Delta_{i_1}x^1 \cdots \Delta_{i_n}x^n} \end{aligned}$$

where $\Delta_j x^k$ is as in (2.2). The next lemma now follows directly using results found in [**3**, pp. 412–415].

LEMMA 1. *Let $g \in C(Y)$ be of bounded variation on all elements of $\tau(I)$ for all $I \in S$. Then if $f_0(p) = \int_{Y(p)} g dv$ for $p \in Y$ we have that*

$$\lim_{N\to\infty} H(\mathcal{P}_N, f_0, f) = \sum_{I\in S_n} \int_I g df$$

*where the convergence is bounded in $N$ for all $f$ in any uniformly bounded subset of $C(Y)$.*

If $F$ is a functional on $C(Y)$ which has a Gaussian distribution with respect to the measure $m$ we will write $F = \mathfrak{N}(\mu, \sigma^2)$ where $\mu$ and $\sigma^2$ are the expectation and variance of $F$.

LEMMA 2. *If $g$ satisfies the conditions of Lemma 1 then the functionals $\{\int_I g df: I \in S\}$ form an independent family and $\int_I g df = \mathfrak{N}(0, \frac{1}{2}\int_I g^2 d\mu_I)$. Moreover, $G(f) = \sum_{I\in S} \int_I g df$ exists for almost all $f$ and $G(f) = \mathfrak{N}(0, \frac{1}{2}\int_Y g^2 dv)$.*

PROOF. If $I \in S$ and $Z_N(f, I)$ is a Riemann-Stieltjes sum for $\int_I g df$ then using the final remark of §2 we see that $Z_N(f, I)$ is a linear combination of independent Gaussian functionals each with zero expectation. Moreover, the variance of $Z_N(f, I)$ is a Riemann sum for $\frac{1}{2}\int_I g^2 d\mu_I$. Hence $\int_I g df = \mathfrak{N}(0, \frac{1}{2}\int_I g^2 d\mu_I)$ for $I \in S$. The independence of the family $\{\int_I g df: I \in S\}$ is also assured because of the independence mentioned at the end of §2. Finally, the distribution of $G(f)$ follows because it is the sum of a sequence of independent Gaussian functionals each with expectation zero and such that the sum of their variances is $\frac{1}{2}\int_Y g^2 dv$ which is finite.

We will denote $\sum_{I\in S}\int_I g df$ by $\int_Y g df$ provided the sum exists for almost all $f$ in $C(Y)$. Let $\{\phi_k(p)\}$ be a complete orthonormal set of functions for $\mathfrak{L}_2(Y)$ each of which satisfies the conditions on $g$ in Lemma 1 for $I \in S$. Let $g$ be in $\mathfrak{L}_2(Y)$ and suppose $g_n(p) = \sum_{k=1}^n c_k\phi_k(p)$ where $c_k = \int_Y g\phi_k dv$. Under these conditions we have the following lemma.

LEMMA 3. *The generalized Stieltjes integral* $\int_Y g d\tilde{f} = \lim_{n\to\infty} \int_Y g_n df$ *exists for almost all* $f \in C(Y)$ *and* $\int_Y g d\tilde{f} = \mathfrak{N}(0, \frac{1}{2}\int_Y g^2 dv)$.

PROOF. Now $\lim_{n\to\infty} \int_Y g_n df = \sum_{k=1}^{\infty} c_k \int_Y \phi_k df$ where $c_k = \int_Y g\phi_k dv$ and $c_k \int_Y \phi_k df = \mathfrak{N}(0, c_k^2/2)$. Furthermore, since $\{\phi_k(p)\}$ is an orthogonal set we see that $\{c_k \int_Y \phi_k df\}$ is an independent family. Therefore, since $\sum_{k=1}^{\infty} c_k^2 = \int_Y g^2 dv$ is finite it follows that

$$\int_Y g d\tilde{f} = \sum_{k=1}^{\infty} c_k \int_Y \phi_k df = \mathfrak{N}\left(0, \frac{1}{2}\int_Y g^2 dv\right).$$

The definition of $\int_Y g d\tilde{f}$ for $g$ in $\mathfrak{L}_2(Y)$ follows that used in [6] to define a generalized Stieltjes integral. Furthermore, if $g$ satisfies the conditions of Lemma 1 for all $I \in S$ it is easily shown that $\int_Y g\, d\tilde{f} = \int_Y g df$ for almost all $f \in C(Y)$. It is also true that $\int_Y g d\tilde{f}$ is essentially independent of the orthonormal basis of $\mathfrak{L}_2(Y)$ used in its definition as long as each element of the basis satisfies the conditions of Lemma 1 for all $I \in S$.

5. **Proof of theorem.** Our proof will first be for the case that $g$ satisfies Lemma 1 and $F$ is a bounded continuous functional which vanishes if $\|f\| = \max_{y\in Y} |f(y)| > M$. Let $J_n$ be the continuous map of $C(Y)$ into $C(Y)$ such that $J_n(f)(x_1, \cdots, x_n, x_{n+1}, \cdots) = f(x_1, \cdots, x_n, \alpha_n)$ for all $(x_1, x_2, \cdots)$ in $Y$ and define $F_n(f) = F(J_n(f))$. Let $\{\mathcal{P}_N\}$ be the sequence of $n$-dimensional rectangular subsets used in Lemma 1 and by $\{G_N\}$ we will mean a corresponding sequence of linear continuous mappings of $C(Y)$ into $C(Y)$ such that $G_N(f)$ depends only on the values of $f$ at points in $\mathcal{P}_N$. We also assume $\lim_{N\to\infty} G_N(f) = f$ on $Y_n$. The construction of a sequence $\{G_N\}$ is obtained through use of the techniques found in [5]. Now using the translation theorem for Lebesgue integrals on $R_{(N+1)^n}$

$$\begin{aligned} E\{F_n(G_N(f))\} &= \exp\{-Q(\mathcal{P}_N, f_0)\} \\ &\quad \cdot E\{F_n(G_N(f+f_0))\exp\{-2H(\mathcal{P}_N, f_0, f)\}\} \end{aligned}$$

where $H(\mathcal{P}_N, f_0, f)$ is defined as in (4.1). Letting $N$ approach infinity and using Lemma 1 along with the bounded convergence theorem we have

$$\begin{aligned} E\{F_n(f)\} &= \exp\left\{-\sum_{I\in S_n} \int_I g^2 d\mu_I\right\} \\ &\quad \cdot E\left\{F_n(f+f_0)\exp\left\{-2\sum_{I\in S_n}\int_I g df\right\}\right\}. \end{aligned} \tag{5.1}$$

Furthermore, using Lemma 2 it follows that

$$\left\{F_n(f+f_0)\exp\left\{-2\sum_{I\in S_n}\int_I g\,df\right\}\right\}$$

is a $\mathfrak{L}_1$ fundamental sequence which converges almost everywhere to $F(f+f_0)\exp\left\{-2\int_Y g\,df\right\}$ and, letting $n$ approach infinity in (5.1), we have that (3.1) holds for $F$ and $g$ as given.

Now let $g$ be in $\mathfrak{L}_2(Y)$ and $\{\phi_k(p)\}$ be an orthonormal basis for $\mathfrak{L}_2(Y)$ such that each $\phi_k$ satisfies Lemma 1 for all $I\in S$. Let $f_n(p) = \int_{Y(p)}\sum_{k=1}^n c_k\phi_k\,d\nu$ where $c_k = \int_Y \phi_k\,g\,d\nu$. Then $\{f_n\}$ converges to $f_0$ in the uniform topology and by the previous case we have for all $n$ that

$$E(F(f)) = \exp\left\{-\int_Y\left(\sum_{k=1}^n c_k\phi_k\right)^2 d\nu\right\}$$
$$\cdot E\left\{F(f+f_n)\exp\left\{-2\sum_{k=1}^n\int_Y c_k\phi_k\,df\right\}\right\}.$$

Since the sequence of functionals integrated above converges almost surely to $F(f+f_0)\exp\left\{-2\int_Y g\,d\bar{f}\right\}$ and it is also $\mathfrak{L}_1$ fundamental we have, by letting $n$ approach infinity, that (3.1) holds for $F$ a bounded continuous functional which vanishes if $\|f\|>M$. The extension to arbitrary measurable $F$ follows as usual so the proof is complete.

## Bibliography

1. R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. of Math. **45** (1944), 386–396.

2. R. H. Cameron and R. E. Graves, *Additive functionals on a space of continuous functions*. I, Trans. Amer. Math. Soc. **70** (1951), 160–176.

3. J. Yeh, *Cameron-Martin translation theorems in the Wiener space of functions of two variables*, Trans. Amer. Math. Soc. **107** (1963), 409–420.

4. M. Marcus and N. A. Khan, *A note on the Hadamard product*, Canad. Math. Bull. **2** (1959), 81–83.

5. J. D. Kuelbs, *Measures on $C(Y)$ when $Y$ is a compact metric space*, Proc. Amer. Math. Soc. **18** (1967), 248–254.

6. R. E. A. C. Paley, N. Wiener and A. Zygmund, *Notes on random functions*, Math. Z. **37** (1933), 647–668.

University of Wisconsin