PROOF. Theorem 3.6 and Example 3.3.

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IMBEDDING CLOSED RIEMANN SURFACES IN C^n

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I. **Introduction.** Let R be a closed Riemann surface of genus g, G a nonempty, open subset of R, and A the set of all complex valued functions that are continuous on R and holomorphic on G. With the usual pointwise operations A is an algebra over the complex field. We consider the problem: how many functions in A suffice to separate points of R?

Let f be a nonconstant member of A. If the genus $g=0$, Wermer [**4**] showed that there exist f_1 and f_2 in A which, together with f separate points of R; if $g=1$, Arens [**2**] established the existence of f_1, f_2 and f_3 in A which, together with f separate points of R. In this note we shall present a modification of the Wermer-Arens argument to prove the following

THEOREM. *Let the genus g be arbitrary. If A contains nonconstant functions, then there exist four functions in A which separate points of R and which have no common branch points in G.*

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II. **Two lemmas.** Let ϕ be a nonconstant member of order n in the field K of meromorphic functions on R. Let w be a point of the extended plane which has n distinct inverse images under ϕ. Denote by $E(\phi, w)$ the finite set which is the union of $\phi^{-1}(w)$ and $\phi^{-1}(\phi(b))$ as b ranges over all the branch points of ϕ. For (fixed) ϕ and ψ in K, let S

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be the set of all z, in R, for which there exists a z' in R with $z \neq z'$, but $\phi(z) = \phi(z')$ and $\psi(z) = \psi(z')$. We then have

LEMMA 1. *Let ψ separate points of $E(\phi, w)$; then S is finite.*

PROOF. There is known (Ahlfors-Sario [**1**, p. 322]) to exist a polynomial $F_v(u) \equiv F(u, v)$ of degree n in u, with rational functions of v as coefficients, which is irreducible in u and is such that $F(\psi(p), \phi(p)) = 0$, $p \in R$. Let $z \in S$, so that $\phi^{-1}(\phi(z))$ consists of n distinct points, say $z_1 = z$, $z_2 = z'$, $z_3 \cdots z_n$. The way the polynomial F is constructed (Ahlfors-Sario [**1**, p. 322]), $u = \psi(z_1)$, $\psi(z_2), \cdots, \psi(z_n)$ constitute the roots of the equation $F(u, \phi(z)) = 0$. Thus, if $z \in S$, this polynomial in u has a repeated root so that its discriminant must be zero. But this discriminant is a rational function $D(v)$ of $v = \phi(z)$ and, since $F(u, v)$ is irreducible in u, it is not identically zero. Hence the set of $\phi(z)$ $(z \in S)$ for which $D(\phi(z)) = 0$ is finite; since ϕ has finite valence n, this implies that S is finite.

For any z in R, let K_z be the set of all those functions in K that are holomorphic on $R \backslash \{z\}$.

LEMMA 2. *If E is a finite subset of $R \backslash \{z\}$, then there exists a member of K_z which separates points of E and which has no branch points in E.*

PROOF. For distinct points z_1, z_2 in $R \backslash \{z\}$ and an integer $n > 2g$, consider the divisors

$$D_1 = z_1 - nz \quad \text{and} \quad D_2 = z_1 + z_2 - nz.$$

By the Riemann-Roch theorem (Ahlfors-Sario [**1**, p. 329]) the dimension, dim D_1, of the complex vector space of those members of K that are multiples of D_1 satisfies

$$\dim D_1 \geqq -\deg D_1 - g + 1 = n - g.$$

Since $\deg D_2 < 2 - 2g$, we have, again by the Riemann-Roch theorem, (Ahlfors-Sario [**1**, p. 329])

$$\dim D_2 = -\deg D_2 - g + 1 = n - g - 1.$$

This and the preceding inequality show that there exists a function belonging to K_z which vanishes at z_1 but not at z_2. Since K_z is an algebra (under the usual operations) over the complex numbers, it follows that there exists ϕ in K_z which separates points of E.

Let $z_1, z_2, \cdots, z_k$ be those points of E at which the multiplicity of $\phi > 1$, and $z_{k+1}, \cdots, z_m$ be the other points of E. Considering, for $n > 2m + 2g - 2$ and $j = 1, \cdots, k$, the divisors $-nz - z_j + \sum_{i=1}^m 2z_i$ and $-nz + \sum_{i=1}^m 2z_i$ and applying the Riemann-Roch theorem as before, one obtains a function ϕ_j in K_z which has a simple zero at z_j and multiple zeros at z_i, $i \neq j$, $i=1, \cdots, m$. Then $\phi + \sum_{j=1}^{k} \phi_j$ serves as a function whose existence is claimed in the lemma.

III. **Proof of the theorem.** Let $p_1 \in G$ and ϕ_1 be a member of K which has its sole pole, of order n_1 say, at p_1. Then, by two applications of Lemmas 1 and 2, choose ϕ_2 and ϕ_3 in K such that ϕ_i has its sole pole, of order n_i say, at $p_i \in G$, $i=2, 3$, $p_i \neq p_j$, for $i \neq j$, $i, j=1, 2, 3$, such that any two of ϕ_1, ϕ_2 and ϕ_3 separate all but a finite number of pairs $(z, z') \in R \times R$ $(z \neq z')$ and such that no two of them have a common branch point. Let E_{ij} be the set of all z in R such that there exists $z' \neq z$ with $\phi_i(z) = \phi_i(z')$ and $\phi_j(z) = \phi_j(z')$. Put $E \equiv \{p_1, p_2, p_3\} \cup_{i<j} E_{ij}$, so that E is a finite subset of R. Suppose that A contains nonconstants functions. Then A separates points of R (Arens [2, p. 255]) and, since A is an algebra, there exists f_0 in A which separates points of E. Set, now, for an integer $n > \max(n_1, n_2, n_3)$,

$$f_i = [f_0 - f_0(p_i)]^n \cdot \phi_i, \qquad i = 1, 2, 3.$$

It is readily seen that $f_1, f_2, f_3 \in A$. To see that f_0, f_1, f_2, and f_3 separate points of R, observe that, if $f_i(z_1) = f_i(z_2)$, $i=0, 1, 2, 3$, then $f_0(p_j) \neq f_0(z_1)$, for at least two values of the index j, say, for $j=1, 2$. Then, we obtain that $\phi_j(z_1) = \phi_j(z_2)$, $j=1, 2$, so that either $z_1 = z_2$ or $z_1, z_2 \in E_{12}$; since f_0 separates points of E_{12} we must indeed have $z_1 = z_2$.

Suppose, if possible, that for some $z \in G$, the differential $df_i(z)$ is zero for $i=0, 1, 2, 3$. Again, we have for two values of j, say, for $j=1, 2$, $f_0(z) \neq f_0(p_j)$, and then we have $d\phi_j(z) = 0$, $j=1, 2$, which is a contradiction to the choice of the ϕ_j. This completes the proof of the theorem.

REMARK. If G is not dense in R, then results of Narasimhan [3] readily imply that three functions in A separate points of R. It is an open problem to determine the minimal number of functions in A that separate points of R.

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