THE FUNCTIONAL EQUATION $f(xy)+f(xy^{-1})=2f(x)f(y)$ FOR GROUPS

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This paper is concerned with the study of the functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)f(y) \tag{A}$$

where f is a complex-valued function on a group G, for all x, y in G. On the line this functional equation is obviously satisfied by the cosine function and may be called the *cosine equation*. Of course this equation has a meaning on any group. One obvious way to solve the functional equation (A) on any group is by means of a homomorphism of G, say g, into the multiplicative group of nonzero complex numbers, K. If g is such a homomorphism, then the function defined by

$$f(x) = \frac{g(x) + g^*(x)}{2}, \quad \text{for all } x \text{ in } G, \tag{B}$$

where $g^*(x)=g(x)^{-1}$, is a solution of the equation (A), as can be seen by an easy calculation.

For $G=R$, the equation (A) is classical, and its continuous solutions are known to be of the form (B) with the continuous g. Recently T. M. Flett [1] found the continuous solutions of (A) on R^2 to be of the form (B), a result extended by the writer to R^n, n any positive integer [**3**]. The question naturally arises as to whether or not *all* solutions of the equation (A) on an arbitrary group have the form (B). In this paper, it is shown that the answer is in the affirmative for Abelian groups and, with a certain restriction, for non-Abelian groups. Furthermore, if f is a continuous solution of (A) on a topological group G, then the corresponding homomorphism g is also continuous. We are indebted to Professor Edwin Hewitt for drawing our attention to the equation (A) and its solutions of the form (B) and for his guidance to the preparations of this paper and to Professor Richard S. Pierce for showing us the greatly simplified versions of our original proofs of Theorems 1 and 2. We are also thankful to the referee for showing us a simplified proof of Theorem 3.

1. THEOREM 1. *Let G be a topological group (Abelian or not) and let f be a complex-valued, continuous function defined on G. Further, let g be a homomorphism of G into K such that $f(x)=(g(x)+g^*(x))/2$, for every x in G. Then g is also continuous.*

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PROOF. By hypothesis,

$$f(xy) = \frac{g(xy) + g^*(xy)}{2}, \quad \text{for all } x, y \text{ in } G$$
$$= \frac{g(x)g(y) + g^*(x)g^*(y)}{2},$$

and

$$f(x)f(y) = \frac{g(x) + g^*(x)}{2} \cdot \frac{g(y) + g^*(y)}{2}$$
$$= \tfrac{1}{4}[g(x)g(y) + g(x)g^*(y) + g^*(x)g(y) + g^*(x)g^*(y)],$$

and so

$$f(xy) - f(x)f(y) = \frac{g(x)g(y) + g^*(x)g^*(y) - g(x)g^*(y) - g^*(x)g(y)}{4}$$
$$(1.1) \qquad = \frac{g(x) - g^*(x)}{2} \cdot \frac{g(y) - g^*(y)}{2}.$$

But,

$$f(x) = \frac{g(x) + g^*(x)}{2},$$

and hence

$$(1.2) \qquad g(x) - f(x) = \frac{g(x) - g^*(x)}{2}.$$

From (1.1) and (1.2), we obtain

$$(1.3) \qquad f(xy) - f(x)f(y) = [g(x) - f(x)][g(y) - f(y)].$$

Case I. Suppose that $g(x) = f(x)$, for every x in G. Then obviously, g is continuous.

Case II. Otherwise, there is an x_0 in G such that $\delta = g(x_0) - f(x_0) \neq 0$. With x_0 for y in (1.3), we get

$$(1.4) \qquad g(x) = f(x) + 1/\delta[f(xx_0) - f(x)f(x_0)].$$

From (1.4), it follows that, if f is continuous, then so is g. This completes the proof of the theorem.

2. From now on, let G be an arbitrary group and let f be a complex-valued function satisfying (A) on G, not identically zero and such that

(C) $$f(xyz) = f(xzy), \quad \text{for every } x, y, z \text{ in } G.$$

We start with some preliminaries which are needed in the sequel. To begin with put $y=e$ in (A), where e is the identity of G. Then we have

$$f(x) + f(x) = 2f(x)f(e)$$

and so

(2.1) $$f(e) = 1.$$

Now setting $x=e$ in (A), we obtain $f(y)+f(y^{-1})=2f(e)f(y)$, from which it follows immediately that,

(2.2) $$f(y^{-1}) = f(y), \quad \text{for every } y \in G.$$

Putting $y=x$ in (A), we get

(2.3) $$f(x^2) + 1 = 2f^2(x), \quad \text{for all } x \in G.$$

We now replace x by xy and y by xy^{-1} in (A). Then by using (C), we obtain that

(2.4)
$$\begin{aligned} 2f(xy)f(xy^{-1}) &= f(xyxy^{-1}) + f(xyyx^{-1}) \\ &= f(xyy^{-1}x) + f(xx^{-1}y^2) \\ &= f(x^2) + f(y^2). \end{aligned}$$

Now using (A), (2.3) and (2.4), we find that

(2.5)
$$\begin{aligned} [f(xy) - f(xy^{-1})]^2 &= [f(xy) + f(xy^{-1})]^2 - 4f(xy)f(xy^{-1}) \\ &= 4f^2(x)f^2(y) - 2[f(x^2) + f(y^2)] \\ &= 4f^2(x)f^2(y) - 2[2f^2(x) - 1 + 2f^2(y) - 1] \\ &= 4[f^2(x)f^2(y) - f^2(x) - f^2(y) + 1] \\ &= 4[f^2(x) - 1][f^2(y) - 1]. \end{aligned}$$

Consequently we obtain from (2.5), that

(2.6) $$f(xy) - f(xy^{-1}) = 2([f^2(x) - 1][f^2(y) - 1])^{1/2},$$

where the square root is unknown. Adding (2.6) and (A), it is easy to see that

(2.7) $$f(xy) = f(x)f(y) + ([f^2(x) - 1][f^2(y) - 1])^{1/2},$$

where again the square root is unknown. From (2.7), we have

(2.8) $$[f(xy) - f(x)f(y)]^2 = [f^2(x) - 1][f^2(y) - 1].$$

3. LEMMA 1. *Let G be any group. Let f be a function on G with the*

properties that (1) f *satisfies* (A) *on* G, (2) $f(x)$ *assumes the values* ± 1 *only on* G, *and* (3) f *satisfies* (C) *on* G. *Then* f *has the form* (B).

PROOF. Since $f^2(x)=1$ for all $x\in G$, (2.7) shows that f is a homomorphism. We also note that $f^*(x)=f(x)$ for all x in G. Thus clearly $f(x)=(f(x)+f^*(x))/2$. This proves the lemma.

4. THEOREM 2. *Let* G *be an arbitrary group. Then every solution of* (A) *on* G *satisfying* (C) *has the form* (B).

PROOF. Let f be a solution of (A) satisfying (C) on G. Lemma 1 is the present theorem if $f(G)\subset\{1, -1\}$. Suppose that there is an x_0 in G such that

$$f^2(x_0) \neq 1. \tag{4.1}$$

Let $\alpha=f(x_0)$ and β be a square root of (α^2-1). That is,

$$\alpha^2 - 1 = \beta^2. \tag{4.2}$$

We now define

$$\begin{aligned} g(x) &= f(x) + 1/\beta[f(xx_0) - f(x)f(x_0)], \qquad \text{for all } x \in G \\ &= 1/\beta[f(xx_0) + (\beta - \alpha)f(x)]. \end{aligned} \tag{4.3}$$

Then g is well defined on G. Further, utilizing (2.3), (4.2) and (4.3), we have

$$\begin{aligned} [g(x) - f(x)]^2 &= 1/\beta^2[f(xx_0) - f(x)f(x_0)]^2 \\ &= 1/\beta^2[f^2(x) - 1][f^2(x_0) - 1] \\ &= (\alpha^2 - 1)/\beta^2[f^2(x) - 1] \\ &= f^2(x) - 1. \end{aligned}$$

Therefore, we obtain

$$g^2(x) - 2g(x)f(x) + 1 = 0. \tag{4.4}$$

From (4.4) we conclude that $g(x) \neq 0$ and moreover $f(x) =(g(x)+g^*(x))/2$ for every x in G. It remains only to prove that g defined by (4.3) is a homomorphism; that is, $g(xy)=g(x)g(y)$, for every x, y in G. With the help of (A) and (C), we obtain

$$\begin{aligned} 2[f(x_0x)f(y) + f(x_0y)f(x)] &= f(x_0xy) + f(x_0xy^{-1}) + f(x_0yx) + f(x_0yx^{-1}) \\ &= f(x_0xy) + f(x_0xy^{-1}) + f(x_0xy) + f(x_0yx^{-1}) \\ &= 2f(x_0xy) + f(x_0xy^{-1}) + f(x_0yx^{-1}) \\ &= 2f(x_0xy) + 2f(x_0)f(xy^{-1}) \\ &= 2[f(x_0xy) + \alpha\{2f(x)f(y) - f(xy)\}]. \end{aligned} \tag{4.5}$$

Again using (A) and (C), we get

$$
\begin{aligned}
2f(x_0x)f(x_0y) &= f(x_0xx_0y) + f(x_0xy^{-1}x_0^{-1}) \\
&= f(x_0x_0yx) + f(x_0x_0^{-1}xy^{-1}) \\
&= f(x_0^2yx) + f(xy^{-1}) \\
&= f(x_0^2xy) + f(xy^{-1}) \qquad (4.6)\\
&= [2f(x_0)f(x_0xy) - f(x_0xyx_0^{-1})] + [2f(x)f(y) - f(xy)] \\
&= [2f(x_0)f(x_0xy) - f(xy)] + [2f(x)f(y) - f(xy)] \\
&= 2[f(x)f(y) + f(x_0)f(x_0xy) - f(xy)].
\end{aligned}
$$

In view of (4.3), (4.5), (4.6) and (4.2), we obtain

$$
\begin{aligned}
g(x)g(y) &= 1/\beta^2[f(x_0x) + (\beta - \alpha)f(x)][f(x_0y) + (\beta - \alpha)f(y)] \\
&= 1/\beta^2[f(x_0x)f(x_0y) + (\beta - \alpha)\{f(x)f(x_0y) + f(y)f(x_0x)\} \\
&\qquad + (\beta - \alpha)^2 f(x)f(y)] \\
&= 1/\beta^2[f(x)f(y) + \alpha f(x_0xy) - f(xy) \\
&\qquad + (\beta - \alpha)\{f(x_0xy) + 2\alpha f(x)f(y) - \alpha f(xy)\} + (\beta - \alpha)^2 f(x)f(y)] \\
&= 1/\beta^2[\{(\beta - \alpha)^2 + 2\alpha(\beta - \alpha) + 1\}f(x)f(y) + \beta f(x_0xy) \\
&\qquad - \{1 + (\beta - \alpha)\alpha\}f(xy)] \\
&= 1/\beta^2[(\beta^2 - \alpha^2 + 1)f(x)f(y) + \beta f(x_0xy) - (\beta\alpha - \beta^2)f(xy)] \\
&= 1/\beta^2[\beta f(x_0xy) + \beta(\beta - \alpha)f(xy)] \\
&= 1/\beta[(\beta - \alpha)f(xy) + f(x_0xy)] \\
&= g(xy).
\end{aligned}
$$

The proof of the theorem is thus complete.

For topological groups we have the following important corollary.

COROLLARY 1. *Let G be a topological group and let f be a continuous solution of* (A) *on G satisfying* (C). *Then f is of the form* (B), *where g is a continuous homomorphism of G into K.*

The proof is immediate from Theorems 1 and 2.

REMARK 1. In Corollary 1, suppose that G is locally compact and that f is (Haar) measurable. Then from (4.3), we see that g is also measurable. But since g is a measurable homomorphism, g is continuous. Hence f is continuous. Therefore every measurable solution of (A) is continuous.

Finally, we observe the following.

5. THEOREM 3. *Let* $(g_1(x)+g_1^*(x))/2=(g_2(x)+g_2^*(x))/2$, *for every*

x in G, where g_1 and g_2 are homomorphisms of G into K. Then either $g_2 = g_1$ or $g_2 = g_1^*$.

PROOF. It is easy to see that

$$[g_1(x)\overset{*}{g_2}(x) - 1][g_1(x)g_2(x) - 1] = 0 \qquad (x \in G). \tag{5.1}$$

The sets G_1 and gG_2 on which the factors in (5.1) vanish are subgroups, since they are kernels of the homomorphisms $g_1g_2^*$ and g_1g_2. From (5.1), $G_1 \cup G_2 = G$. But a group is never the union of two subgroups for, if $x_1 \notin G_1$ and $x_2 \notin G_2$, then one of x_1, x_2, x_1x_2 is not in $G_1 \cup G_2$. Therefore $G_1 = G$ or $G_2 = G$ that is, $g_2 = g_1$ or $g_2 = g_1^*$. This proves the present theorem.

Our Theorem (2) permits us to compute in detail all continuous solutions of (A) on an arbitrary locally compact Abelian group G. The homomorphisms g of (B) all have the form $\chi\exp(\psi)$, where χ is a continuous character of G and ψ is a continuous real character of G (in the sense of [**2**, p. 389]). The group G has the form $R^n\chi G_0$, where G_0 contains a compact subgroup (open) J_0, where n is any positive integer [**2**, p. 389]. The form of ψ on R^n is obvious, and ψ is identically zero on every element of G some power of which lies in J_0. Otherwise ψ on G_0 is easily described by a Zorn's Lemma argument, since the image group R of ψ is divisible.

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