

INVERSE LIMITS OF PERFECTLY NORMAL SPACES

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It is the purpose of this note to establish that the countable inverse limit of perfectly normal topological spaces is again a perfectly normal space and to establish some of the consequences of that fact as it pertains to Moore spaces. In particular, this result yields a new proof of the fact that, if there is a normal nonmetrizable (complete) Moore space, then there is one which is not locally metrizable at any point [2], [6].

Throughout this note, $\text{Cl}(M)$ denotes the closure of the point set M . If $\{X_n, \pi_n^m\}$ is an inverse mapping system with inverse limit X_∞ and n is a positive integer, π_n denotes the projection of X_∞ into X_n (all inverse mapping systems are taken over the set of positive integers directed by $<$).

The authors are indebted to participants of the Arizona State University Conference on Point Set Topology, March 1967, particularly Professor E. Michael, for probing questions which led us to obtain for, and include in, this note more general results than had been planned. We are also indebted to the referee for valuable suggestions.

THEOREM 1. *If X_∞ is the inverse limit of the inverse mapping system $\{X_n, \pi_n^m\}$ where, for each n , X_n is a topological space in which each closed set is an inner limiting ($=G_\delta$) set and π_n^{n+1} is a mapping of X_{n+1} into X_n , then each closed subset of X_∞ is an inner limiting set.*

PROOF. Suppose that $\{X_n, \pi_n^m\}$ and X_∞ are as in the hypothesis and M is a closed subset of X_∞ . There exist sequences $0_{11}, 0_{12}, \dots; 0_{21}, 0_{22}, \dots; \dots$ such that, for each integer n , $0_{n1}, 0_{n2}, \dots$ is a sequence of domains whose common part is $\text{Cl}(\pi_n(M))$ and, for each positive integer m , 0_{nm} contains $0_{n,m+1}$ and $(\pi_n^{n+1})^{-1}(0_{nm})$ contains $0_{n+1,m}$. Clearly, for each n , $\pi_n^{-1}(0_{nn})$ contains M . Suppose that p is a point of X_∞ not in M such that, for each n , $\pi_n^{-1}(0_{nn})$ contains p . For some positive integer n , $\pi_n(p)$ is not in $\text{Cl}(\pi_n(M))$ and, hence, for some $m > n$, $\pi_n(p)$ is not in 0_{nm} . Then $\pi_m(p)$ is not in 0_{mm} , which is a subset of $(\pi_n^m)^{-1}(0_{nm})$. Thus p is not in $\pi_m^{-1}(0_{mm})$ and M is an inner limiting set.

THEOREM 2. *If X_∞ is the inverse limit of the inverse mapping system $\{X_n, \pi_n^m\}$ where, for each n , X_n is a perfectly normal space and π_n^{n+1} is a mapping of X_{n+1} into X_n , then X_∞ is perfectly normal.*

Received by the editors October 31, 1966.

PROOF. Let $\{X_n, \pi_n^m\}$ and X_∞ be as in the hypothesis and let H and K be mutually exclusive closed point sets in X_∞ . It follows from Theorem 1 that every closed set in X_∞ is an inner limiting set.

There exist sequences $0_{11}, 0_{12}, \dots; 0_{21}, 0_{22}, \dots; \dots$ such that, for each n , $0_{n1}, 0_{n2}, \dots$ is a sequence of domains in X_n whose common part is $\text{Cl}(\pi_n(H)) \cdot \text{Cl}(\pi_n(K))$ and, for each m , 0_{nm} contains $\text{Cl}(0_{n,m+1})$ and $(\pi_n^{m+1})^{-1}(0_{nm})$ contains $\text{Cl}(0_{n+1,m})$. Now, $\text{Cl}(\pi_1(H)) \setminus 0_{11}$ and $\text{Cl}(\pi_1(K)) \setminus 0_{11}$ are two mutually exclusive closed point sets in X_1 , the former does not intersect $\text{Cl}(\pi_1(K))$, and the latter does not intersect $\text{Cl}(\pi_1(H))$. Therefore, there exist in X_1 domains D_1 and E_1 containing $\text{Cl}(\pi_1(H)) \setminus 0_{11}$ and $\text{Cl}(\pi_1(K)) \setminus 0_{11}$ respectively such that $\text{Cl}(D_1)$ does not intersect $\text{Cl}(E_1)$ or $\text{Cl}(\pi_1(K))$ and $\text{Cl}(E_1)$ does not intersect $\text{Cl}(\pi_1(H))$. Similarly, there exist in X_2 domains D_2 and E_2 containing $(\pi_1^2)^{-1}(\text{Cl}(D_1)) + [\text{Cl}(\pi_2(H)) \setminus 0_{22}]$ and $(\pi_1^2)^{-1}(\text{Cl}(E_1)) + [\text{Cl}(\pi_2(K)) \setminus 0_{22}]$ respectively such that $\text{Cl}(D_2)$ does not intersect $\text{Cl}(E_2)$ or $(\pi_1^2)^{-1}(\text{Cl}(E_1)) + \text{Cl}(\pi_2(K))$ and $\text{Cl}(E_2)$ does not intersect $(\pi_1^2)^{-1}(\text{Cl}(E_1)) + \text{Cl}(\pi_2(H))$. This process may be continued to obtain sequences D_1, D_2, D_3, \dots and E_1, E_2, E_3, \dots such that, for each positive integer n ,

- (i) D_n and E_n are mutually exclusive domains in X_n ;
 - (ii) $\pi_n^{-1}(D_n)$ is a subset of $\pi_{n+1}^{-1}(D_{n+1})$ and $\pi_n^{-1}(E_n)$ is a subset of $\pi_{n+1}^{-1}(E_{n+1})$; and
 - (iii) D_n contains $\text{Cl}(\pi_n(H)) \setminus 0_{nn}$ and E_n contains $\text{Cl}(\pi_n(K)) \setminus 0_{nn}$.
- Let $D = \pi_1^{-1}(D_1) + \pi_2^{-1}(D_2) + \dots$ and $E = \pi_1^{-1}(E_1) + \pi_2^{-1}(E_2) + \dots$.

It follows from (i) and (ii) that D and E are mutually exclusive domains in X_∞ . It remains to be proved that D contains H and E contains K .

Suppose that p is a point of H . Then p is not in K and there exists a positive integer n such that $\pi_n(p)$ is not in $\text{Cl}(\pi_n(K))$. There is a positive integer $m > n$ such that $\pi_n(p)$ is not in 0_{nm} . Now, $\pi_m(p)$ is not in $(\pi_n^m)^{-1}(0_{nm})$ and is, therefore, not in 0_{mm} . Thus $\pi_m(p)$ is in D_m and, hence, p is in $\pi_m^{-1}(D_m)$. Similarly, E contains K .

The following Theorem of Katetov [3] is a corollary of Theorem 2.

THEOREM OF KATETOV. *If all spaces $P_1 \times \dots \times P_n$ ($n = 1, 2, \dots$) are perfectly normal, then the space $P = P_1 \times P_2 \times \dots$ is perfectly normal as well.*

PROOF. For each n , let π_n^{n+1} be the projection of $P_1 \times \dots \times P_{n+1}$ onto $P_1 \times \dots \times P_n$. Then the inverse limit of the inverse mapping system so obtained is topologically equivalent to P and is perfectly normal.

REMARK. Katetov has shown, [3], by an example that his above

stated theorem does not hold if perfectly normal is replaced by completely normal. Therefore, our Theorem 2 does not hold if perfectly normal is replaced by completely normal even if all of the bonding mappings are onto.

THEOREM 3. *If X_∞ is the inverse limit of the inverse mapping system $\{X_n, \pi_n^m\}$ where, for each n , X_n is a (complete) normal Moore space and π_n^{n+1} is a mapping of X_{n+1} into X_n , then X_∞ is a (complete) normal Moore space.*

PROOF. It is known (see, for example, [1] and [7]) that the Cartesian product of countably many (complete) Moore spaces is a (complete) Moore space. Every subspace of a Moore space is a Moore space and every closed subspace of a complete Moore space is complete, [5]. Thus, the countable inverse limit of Moore spaces is a Moore space and, since the inverse limit is closed in the Cartesian product space [4, p. 31], it is complete if the coordinate spaces are complete. Now, each normal Moore space is perfectly normal and the Theorem is proved.

COROLLARY 1. *If each Cartesian product of two normal Moore spaces is normal, then so is each Cartesian product of countably infinitely many.*

The proof of this Corollary is precisely the same as that for the above Theorem of Katetov, or, as has been pointed out by Bruce Anderson, [1], it may be noted that this corollary follows from the Theorem of Katetov.

D. R. Traylor and the second author of this paper showed [2] that, if S^0 is a Moore space, there is a Moore space S^w such that every open set in S^w contains a copy of S^0 and S^w is normal if S^0 is. Traylor showed [6] that S^w is completable in such a way that the completion of S^w is normal if S^0 is complete and normal. The proofs of these results are simplified considerably by using Theorem 3 in the following:

COROLLARY 2. *If S^0 is a Moore space then there is a Moore space S_∞ in which every open set contains a topological copy of S^0 and S_∞ is complete or normal, respectively, if S^0 is complete or normal.*

PROOF. Let S^0 be a Moore space. The notation of [2] will be used. Let M^j , $S_{p,k}^j$, and S^j be as in [2], i.e., (1) M^j is a dense subset of S^j , (2) $S_{p,1}^{j+1}, S_{p,2}^{j+1}, \dots$ is a null sequence, of copies of S^0 , which converges in S^{j+1} to the point P of M^j , and (3) $S^{j+1} = S^j + \sum_{P \in M^j} \sum_{k=1}^{\infty} S_{p,k}^{j+1}$. Define π_j^{j+1} by $\pi_j^{j+1}(x) = x$ if x is in S^j and $\pi_j^{j+1}(x) = P$ if there is a positive integer k such that x is in $S_{p,k}^{j+1}$. Then π_j^{j+1} is a continuous

transformation of S^{j+1} onto S^j . The inverse limit, S_∞ , of the inverse mapping system $\{S^n, \pi_n^m\}$ is the desired space.

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