SYMMETRIZING KERNELS AND THE INTEGRAL EQUATIONS OF FIRST KIND OF CLASSICAL POTENTIAL THEORY

JAMES LUCIEN HOWLAND

1. A symmetric kernel G is said to symmetrize the kernel K by composition on the left in case the product GK is symmetric—i.e. in case $GK = K^TG$. It follows at once that, if G is a left symmetrizer of K, so are GK, GK^2, GK^3, etc., and that the linear manifold spanned by these kernels consists entirely of left symmetrizers of K. It also follows that G^{-1}, when it exists, satisfies $KG^{-1} = G^{-1}K^T$, so that the inverse of G is to be sought among the right symmetrizers of K.

The integral equations of first kind of classical potential theory, namely

$$f(p) = \int_S G(pq)\mu(q)dS_q, \tag{1}$$

$$g(p) = \int_S D(pq)\nu(q)dS_q \tag{2}$$

arise when the solution of the Dirichlet problem with respect to a surface S is sought in the form of the potential $V[\mu]$ of a surface distribution on S of density μ, and the solution of the Neumann problem is sought in the form of the potential $W[\nu]$ of a double layer on S of moment ν. In this notation, $G(p, q) = 1/(2\pi r_{pq})$ is the potential at $p(q)$ of a unit mass at $q(p)$, and is symmetric, while $D(p, q) = (\partial^2/\partial n_p \partial n_q)G(pq)$ represents the normal component of force at $p(q)$ due to a unit normal dipole at $q(p)$ and is likewise symmetric. The given boundary values relevant to the (interior or exterior) Dirichlet and Neumann problems are $f(p)$ and $g(p)$, respectively.

In this paper, the concepts of the first paragraph above are applied to the solution of the equations (1) and (2) in the case of a closed, bounded surface S of class B [**11**, p. 186]. For, it is known [**19**, §4, p. 344] that G is a left symmetrizer of the kernel

$$K(p, q) = (\partial/\partial n_p)G(p, q) = \cos(r, n_p)/(2\pi r_{pq}^2)$$

of the Fredholm-Poincaré integral equations. It will be shown that

$$D(p, q) = [\cos(n_p, n_q) + 3\cos(n_p, r)\cos(n_q, r)]/(2\pi r_{pq}^3)$$

Presented to the Society, November 25, 1966 under the title *Symmetrizing kernels and the integral equations of first kind of potential theory*; received by the editors November 8, 1966.

is a right symmetrizer of K and that $DG=K^2-I$ so that, in effect

$$G^{-1}=-[D+K^2D+K^4D+\cdots], \qquad D^{-1}=-[G+GK^2+GK^4+\cdots]$$

and the integral equations (1) and (2) are equivalent, respectively, to

$$K^2\mu-\mu=Df, \tag{3}$$

$$(K^T)^2\nu-\nu=Gg. \tag{4}$$

Analogous results may be obtained in the plane.

The problem represented by the integral equation (1) has been discussed by Liapounoff [10] who showed that, when f is such that $W[f]$ admits a regular normal derivative on S, it may be written $f=G\mu$. The density μ is obtained as the difference of the solutions of two integral equations, formulated with respect to the regions interior and exterior to S, respectively. For general f he showed that the third and succeeding terms of the Neumann series solution of the Dirichlet problem with boundary values f could be written as the potential of a single layer but, in general, the first two terms could not. Similar results were obtained by E. R. Neumann [14] who, in addition, obtained the solution of the Neumann problem in the form of a double layer potential [14, pp. 43–66]—i.e. solved the integral equation (2). Bertrand [1] converted the equation (1), in the two-dimensional case, to an equation of second kind by differentiation while Plume [21] has given a similar treatment of the Neumann problem. Picard [17], [18], in a well-known paper, has given necessary and sufficient conditions, applicable in the two-dimensional case, that (1) admit a square-integrable solution, and has worked out the case of a circle. These methods have been extended to the three-dimensional case by Fenyo [3], who illustrates his results in the case of a sphere. Blumer [2] converts (1), in the three-dimensional case, to each of three integral equations of second kind by a complicated process based upon integro-differential operators analogous to those of M. Riesz. Thus it appears that the symmetrizing property of D and the equivalence of (1) and (2) with equations of second kind with kernels K^2 and $(K^T)^2$ are new results. The following development, however, owes much to the work of Liapounoff [8], [9], [10].

2. A function $V(P)$ harmonic in the region R interior, or R' exterior, to S is said to possess a regular normal derivative on S [9, §2, p. 246, §16, p. 285] in case $\lim_{P\to p\in S}(\partial V/\partial n)(P)$ is taken uniformly on S as $P\to p$ along the normal to S at p. (The normal n is defined throughout as the interior normal to S, and determines the positive (interior) and negative (exterior) sides of S. See, e.g. equations (6)

and (8).) These limiting values then define a continuous function on S. Tauber [22], [23] (see also Liapounoff [8, p. 131]) has shown that the difference of the derivatives of a double layer potential $W[\nu]$, with continuous moment ν on S, in the normal direction at points on the normal on opposite sides and equidistant from S, vanishes as these points approach S, and hence that, if $W[\nu]$ admits a regular normal derivative on one side of S, it does so on the other side, and the limiting values of the normal derivatives are equal. Gunther [6, p. 70] has quoted an example of a surface S and continuous function ν such that $W[\nu]$ does not admit a regular normal derivative. Analytic conditions sufficient for the existence of a regular normal derivative $D\nu$ of $W[\nu]$ have been formulated by C. Neumann [12, p. 413], [13, p. 436], Liapounoff [8, p. 132], [9, §20, p. 293 et seq.], and Kellogg [7, p. 42 et seq.] while complicated necessary and sufficient conditions have been established by Petrini [15, p. 320], [16, p. 212]. Liapounoff [9, §19, p. 293] has characterized the domain of D by showing that, for continuous ν, $W[\nu]$ admits a regular normal derivative on S when, and only when, the solution of the Dirichlet problem with respect to R, determined by the boundary values ν, also admits a regular normal derivative on S.

Liapounoff [9, §§15, 16] (see also Plemelj [20, §4, p. 9]) has established the following extension of Green's third identity:

LEMMA. *Suppose that $V(P)$ is harmonic in R and admits a regular normal derivative on S. Then, when $P \in R$,*

$$\chi(P) = \tfrac{1}{2}W[V] - \tfrac{1}{2}V[\partial V/\partial n] = V(P) \tag{5}$$

and, when $P \in R'$, $\chi(P) = 0$.

It follows at once from (5), together with the formulae

$$W\,|_{+} = \nu + K^T\nu, \qquad W\,|_{-} = -\nu + K^T\nu, \tag{6}$$

describing the discontinuity in $W[\nu]$ across S, that, on S,

$$V = K^TV - G(\partial V/\partial n). \tag{7}$$

Moreover, if it is assumed that $W[V]$ admits a regular normal derivative on S, it follows from (5) and the formulae

$$(\partial V/\partial n)\,|_{+} = -\mu + K\mu, \qquad (\partial V/\partial n)\,|_{-} = \mu + K\mu \tag{8}$$

describing the discontinuity in the normal derivative of $V[\mu]$ across S that, on S,

$$\partial V/\partial n = DV - K(\partial V/\partial n). \tag{9}$$

The formulae (7) and (9) may be interpreted as integral equations connecting the limiting values V and $\partial V/\partial n$ on S. It is important to observe that (7) is the Neumann-Poincaré integral equation in the case $\lambda=+1$, corresponding to the exterior Dirichlet problem, while (9) is the Robin-Poincaré integral equation in the case $\lambda=-1$, corresponding to the exterior Neumann problem. The questions of existence and uniqueness of solutions of these equations have been discussed in detail by Plemelj [**19**, §16, p. 383 et seq.]. In particular, it is known that $\lambda=-1$ is not an eigenvalue of K and that (9) possesses an unique, continuous solution for any nonhomogeneous term DV.

3. The characteristic properties of G and D now follow at once. For, whenever μ is continuous on S, substitution from the first of equations (8) into (7) is permissible and leads directly to the formula (see also Plemelj, loc. cit.)

$$GK\mu = K^T G\mu. \tag{10}$$

Similarly, since $V[\mu]$ admits a regular normal derivative on S, so does $W[V]$, and substitution from the first of equations (8) into (9) is also permissible, to obtain

$$DG\mu = K^2\mu - \mu. \tag{11}$$

When ν is continuous on S, $W[\nu]$ may be represented in R as the sum $W[\nu]=V_1(P)+V_2(P)$ of two harmonic functions characterized by the boundary values ν and $K^T\nu$ on S, respectively. When $W[\nu]$ admits a regular normal derivative on S, so does $V_1(P)$, whence $V_2(P)$ does also, and so $W[K^T\nu]$ does also. Thus D and $DK^T\nu$ both exist, and $D(\nu+K^T\nu)=D\nu+DK^T\nu$. Substituting, then, from the first of equations (6) into (9), and applying this relation, the formula

$$DK^T\nu = KD\nu \tag{12}$$

is obtained. Similar substitution into (7) leads to

$$GD\nu = (K^T)^2\nu - \nu. \tag{13}$$

4. It is well known (Plemelj, loc. cit. §2) that $\lambda=+1$ is an eigenvalue of the kernel K of the Fredholm-Poincaré integral equations, and that, correspondingly, the homogeneous equations $K^T\nu_1-\nu_1=0$, $K\mu_1-\mu_1=0$ each admit a single eigenfunction. The eigenfunction ν_1 is constant, while μ_1 represents the equilibrium distribution of charge on S. It follows from (10) that, with appropriate normalization, $\nu_1=G\mu_1$. On the other hand, since μ_1 is continuous, $V[\mu_1]$ has a regular normal derivative on S whence, $W[V]=W[\nu_1]$ has also. However, $D\nu_1=0\neq\mu_1$.

The identities $K^2\mu-\mu=K\mu-\mu+K(K\mu-\mu)$ and $(K^T)^2\nu-\nu=K^T\nu-\nu+K^T(K^T\nu-\nu)$, together with the fact that $\lambda=-1$ is not an eigenvalue of K or K^T, show that μ_1 and ν_1 are also the only eigenfunctions of K^2 and $(K^T)^2$ respectively. Thus, it follows from (13) that $D\nu=0$ implies $\nu=$constant.

These remarks, together with Fredholm's third theorem, show that the integral equations (3) and (4) admit solutions when, and only when, $\int DfdS=0$ and $\int gG\mu_1 dS=0$, respectively; and that these solutions are not unique but contain an added arbitrary multiple of the corresponding eigenfunction.

5. THEOREM 1. *A necessary and sufficient condition that the integral equation* (1) *shall admit an unique continuous solution μ for any given continuous function f is that f lie in the domain of D. When this is the case, μ satisfies the equation* (3).

PROOF. 1. When μ is a continuous solution of (1), $V[\mu]$ admits a regular normal derivative on S and, thus, so does $W[f]$. From (8)and (9) it follows that μ satisfies (3).

2. When Df exists, equation (3) may be formulated and, since $\int_S DfdS=0$ this equation admits a continuous solution $\mu=\mu_0+C\mu_1$. For each such solution it follows from (1) that $DG\mu=Df$, whence $G\mu-f=G\mu_0+CG\mu_1-f$ is constant on S. But $G\mu_1=\nu_1$ is itself constant, thus C may be uniquely chosen such that (1) is satisfied. Q.E.D.

THEOREM 2. *A necessary and sufficient condition that the integral equation* (2) *shall admit a continuous solution ν, unique to within an additive constant, for any given continuous function g, is*

$$\int_S gdS = 0. \tag{14}$$

When this is the case, ν satisfies the equation (4) *or, alternatively, $\nu=G\mu$ where μ satisfies the equation* (3) *with Df replaced with g.*

PROOF. 1. When ν_0 satisfies (2), so that $D\nu_0=g=\partial/\partial n W[\nu_0]$ then g satisfies (14). Moreover, from (13), $GD\nu_0=Gg=(K^T)^2\nu_0-\nu_0$, whence ν_0 satisfies (4). Since ν_0 lies in the domain of D, $\nu_0=G\mu$ for some continuous μ, and μ satisfies (3) with Df replaced with g, by Theorem 1. These same conclusions are clearly valid for $\nu=\nu_0+C\nu_1$, which also satisfies (2).

2. Given a continuous function g satisfying (14), it follows that $\int\mu_1 GgdS=\int\mu_1 gdS=\nu_1\int gdS=0$, whence (4) possesses a continuous solution $\nu=\nu_0+C\nu_1$. Similarly (3) with Df replaced with g possesses a continuous solution $\mu=\mu_0+C\mu_1$, whence $G\mu=G\mu_0+CG\mu_1=G\mu_0+C\nu_1$

and it follows from (10) that $\nu = G\mu$ and that every solution of (4) has this form. Thus, $W[\nu]$ has a regular normal derivative on S, and from (13) it follows that $GD\nu = Gf$, whence $D\nu = f$. Q.E.D.

Since $W[\nu_1]$ is constant in R and zero in R' this theorem is in accordance with the known fact that the Neumann problem possesses an unique, regular solution in R', but that the solution is only determined to within an additive constant in R.

A second integral equation may, in certain circumstances, be formulated for the Dirichlet problem as follows:

COROLLARY. *Suppose that* $\int f\mu_1 dS = 0$. *Then, the solution* μ *of* (1) *and* (3) *may be written* $\mu = D\nu$ *where* ν *is a continuous solution of* (4) *with* Gg *replaced with* f.

PROOF. 1. When μ satisfies (1), $0 = \int \mu_1 f dS = \int \mu_1 G\mu dS = \int \nu_1 \mu dS = \nu_1 \int \mu dS$ so that, by Theorem 2, $D\nu = \mu$ possesses a continuous solution ν. This function satisfies (4) with Gg replaced with f.

2. Following the arguments of Theorem 2, it is seen that (4), with Gg replaced with f, possesses continuous solutions ν for which $W[\nu]$ admits an unique normal derivative on S. From (13), $GD\nu = f$ whence $\mu = D\nu$ is the solution of (1). Q.E.D.

REFERENCES

1. G. Bertrand, *Le problème de Dirichlet et le potentiel de simple couche*, Bull. Sci. Math. (2) **47** (1923), 282–288, 298–307.

2. H. Blumer, *Beziehungen zwischen speziellen linearen Integralgleichungen erster und zweiter Art und Lösung des Dirichletschen Problems durch das Potential einer einfachen Schicht*, Comment Math. Helv. **28** (1954), 197–224.

3. S. Fenyo, *Über das Dirichletsche Problem bezüglich der Kugel*, Publ. Math. Debrecen **3** (1953), 71–80.

4. I. Fredholm, *Sur une nouvelle méthode pour la résolution du problème de Dirichlet*, Öfversigt Kungl. Vetenskapsakad. Stockholm (1900), 39–46.

5. ———, *Sur une classe d'equations fonctionelles*, Acta Math. **27** (1903), 365–390.

6. N. Gunther, *La théorie du potentiel et ses applications aux problèmes fondamentaux de la physique mathématique*, Gauthier-Villars, Paris, 1934.

7. O. Kellogg, *Potential functions on the boundary of their regions of definition*, Trans. Amer. Math. Soc. **9** (1908), 39–50.

8. A. Liapounoff, *Sur le potentiel de la double couche*, Comm. Soc. Math. Kharkov **6** (1897), 129–138.

9. ———, *Sur certaines questions qui se rattachent au problème de Dirichlet*, J. Math. Pures Appl. (5) **4** (1898), 3, 241–311.

10. ———, *Sur le principe fondamental de la méthode de Neumann dans le problème de Dirichlet*, Comm. Soc. Math. Kharkov (2) **7** (1902), 229–252.

11. L. Lichtenstein, *Neure Entwicklung der Potentieltheorie, Konforme Abbildung.* Encyklop. math. Wiss. Vol. II C **3** (1918), 177–377.

12. C. Neumann, *Neue Sätze über das Logarithmische Potential*, Math. Ann. **16** (1882), 409–431.

13. ———, *Neue Sätze das Newton' sche Potential*, Math. Ann. **16** (1882), 432–438.

14. E. Neumann, *Beitrage zu einzelen Fragen der höheren Potentialtheorie*, Teubner, Leipzig, 1912.

15. H. Petrini, *Les derivées premières et secondes du potentiel*, Acta Math. **31** (1908), 127–332.

16. ———, *Les derivées premières et secondes du potentiel logarithmique*, J. Math. Pures Appl. (6) **5** (1909), 127–223.

17. E. Picard, *Quelques remarques sur les équations intégrales de première éspèce et sur certains problèmes de physique mathématique*, C. R. Acad. Sci. Paris **148** (1909), 1563–1568.

18. ———, *Sur un théorème général relatif aux équations intégrales de première éspèce et sur quelques problèmes de physique mathématique*, Rend. Circ. Mat. Palermo **29** (1910), 79–97.

19. J. Plemelj, *Über Lineare Randwertaufgaben der Potentialtheorie*, Monatsh. Mat. Physik **15** (1904), 337–411.

20. ———, *Potentialtheoretische Untersuchungen*, Teubner, Leipzig, 1911.

21. Z. Plume, *Potencialteorijas neimana problemas atrisināšana ar singularo integralvienadojumu metodi*, Latvijas PSR Zinātņu Akad. Vēstis (37) **8** (1950), 119–126.

22. A. Tauber, *Über das Potential einer Doppelbelegung*, Monatsh. Math. Physik **8** (1897), 79–86.

23. ——— *Über Einige Sätze der Potentialtheorie*, Monatsh. Math. Physik **9** (1898), 74–88.

UNIVERSITY OF OTTAWA