TITCHMARSH'S CONVOLUTION THEOREM ON GROUPS

BENJAMIN WEISS

There is a well-known theorem of Titchmarsh concerning measures with compact support which may be stated as follows.

(T) If μ, ν are finite measures on $\boldsymbol{R}$ (the real line) with compact support, and their intervals of support are $[a, b]$, $[c, d]$ resp. then the interval of support of $\mu * \nu$ is $[a+c, b+d]$.

Recall that the interval of support of a measure is the *smallest* interval which contains the support of the measure. Note that the fact that $\operatorname{supp}(\mu * \nu) \subset [a+c, b+d]$ is trivial, what the theorem asserts is that $\operatorname{supp}(\mu * \nu)$ is contained in no smaller interval. The theorem was extended by Lions [2] to finite measures on $\boldsymbol{R}^n$ with the convex closure of the support of a measure replacing the interval of support. In the following, we take up the question of characterizing those locally compact abelian groups for which an analogue of (T) holds. Our proof, like Lions', relies on a reduction to the one dimensional theorem, but unlike his makes only minimal use[1] of complex variable methods, and thus is in the spirit of the "real variable" proof of Ryll-Nardzewski [4] or the functional analytic proof of Kalisch [1] for (T).

We begin with a very special case of (T), and investigate its validity for general l.c.a. groups G. The special case is

(ST) If μ and ν are finite measures on G with compact support and $\mu * \nu = 0$ then either $\mu = 0$ or $\nu = 0$.

If G is a compact group, and μ denotes Haar measure, set $\nu = \hat{t}\mu$ where $\hat{t}$ is any character of G; then it is easily checked that $\mu * \nu = 0$, while both are nonzero with compact support. Thus (ST) fails to hold if G is compact. In the same way one sees that (ST) doesn't hold if G contains a nontrivial compact subgroup. The remarkable fact is that if G doesn't contain any nontrivial compact subgroups (ST) holds and what is more so does an appropriate version of (T).

Denote the dual of G by $\hat{G}$ and let $\hat{G}_e$ be the connected component of the identity in $\hat{G}$. If $\hat{G}_e$ is not all of $\hat{G}$ there is a nontrivial subgroup H of $\hat{G}$ containing $\hat{G}_e$ such that $\hat{G}/H$ is discrete [6]. The duality theorem gives a nontrivial compact subgroup of G; and thus G possessing no compact subgroups implies $\hat{G}$ connected. Recall that a real character x of G is a continuous homomorphism of G into the additive

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[1] Cf. Lemma.

group of real numbers. The vector space of real characters of G will be denoted by X. Every $x \in X$ gives rise to a one parameter subgroup $\hat{t}_s$ of $\hat{G}$ defined by

$$\hat{t}_s(t) = \exp\{\text{is } x(t)\}.$$

It is known [3] that the connectivity of $\hat{G}$ is equivalent with the statement that these one parameter subgroups are dense in $\hat{G}$, and it is this latter characterization that we will need.

If μ is a finite measure on G, and $x \in X$, we define the measure μ_x on R by

$$\int_{-\infty}^{\infty} g(r)d\mu_x(r) = \int_G g(x(t))d\mu(t).$$

Specializing g to the indicator function of $[a, b]$ we have

$$\mu_x([a, b]) = \mu(\{t: a \leqq x(t) \leqq b\}).$$

For positive measures in $\boldsymbol{R}^n$, where X coincides with $\boldsymbol{R}^n$, the μ_x are just the well-known one dimensional marginal distributions of μ. The following lemma will be needed.

LEMMA. *Let G be a l.c.a.g. with connected dual $\hat{G}$, μ a finite measure with compact support, and $\hat{\mu}$ its Fourier transform which is defined by*

$$\hat{\mu}(\hat{t}) = \int_G \hat{t}(t)d\mu(t).$$

Then if $\hat{\mu}$ vanishes on an open set in $\hat{G}$ $\mu = 0$.

PROOF. If μ vanishes in a neighborhood of $\hat{t}_0$, then we replace μ by $\hat{t}_0\mu$; so that we may assume that $\hat{\mu}$ vanishes in a neighborhood of the identity. For any real character x, we form

$$\int_G \exp(ux(t) + ivx(t))d\mu(t) = \int_{-\infty}^{\infty} \exp(r(u + iv))d\mu_x(r) = f(u + iv).$$

Since μ has compact support so does μ_x and hence f is an analytic function of $z = u + iv$. But for $u = 0$ and v in a neighborhood of the identity f vanishes and thus f is identically zero. We conclude that $\hat{\mu}$ vanishes on the entire one-parameter subgroup of $\hat{G}$ defined by x. Since $\hat{G}$ is connected these subgroups are dense and this together with the continuity of $\hat{\mu}$ implies that $\hat{\mu} = 0$ which in turn implies that $\mu = 0$.

A convex set in R^n can be described by giving, for each direction, the signed distances from the origin of its supporting hyperplanes. The natural generalization of this to G is given by the following:

$$\overset{+}{s}_x(A) = \sup_{t\in A} x(t), \qquad \overset{-}{s}_x(A) = \inf_{t\in A} x(t)$$

where $x\in X$, $A\subset G$. We admit the values $\pm\infty$. The convex hull of a set A is then defined by

$$\operatorname{co}(A) = \bigcap_{x\in X} \left\{t: \overset{-}{s}_x(A) \leqq x(t) \leqq \overset{+}{s}_x(A)\right\}.$$

If $G=R$, we denote by $s^+(A)$, $s^-(A)$ the right and left end points of the convex hull of $A\subset R$.

If μ is a finite measure on G, and $\mathcal{S}(\mu)$ its support we write

$$\overset{\pm}{s}_x(\mu) = \overset{\pm}{s}_x(\mathcal{S}(\mu)).$$

It is easy to verify

$$\overset{\pm}{s}_x(A+B) = \overset{\pm}{s}_x(A) + \overset{\pm}{s}_x(B)$$

and hence

$$\operatorname{co}(A+B) = \operatorname{co}(A) + \operatorname{co}(B).$$

The connection between μ_x and μ leads to

$$(1) \qquad \overset{-}{s}_x(\mu) \leqq s^-(\mu_x) \leqq s^+(\mu_x) \leqq \overset{+}{s}_x(\mu),$$

where strict inequalities may occur. The Titchmarsh theorem for G may be stated as follows.

THEOREM. *If μ, ν are finite measures on G with compact support; and G has no compact subgroups then*

$$\operatorname{co}(\mathcal{S}(\mu * \nu)) = \operatorname{co}(\mathcal{S}(\mu)) + \operatorname{co}(\mathcal{S}(\nu)).$$

PROOF. By the preceding remarks it suffices to show that for any real character x

$$\overset{\pm}{s}_x(\mu * \nu) = \overset{\pm}{s}_x(\mu) + \overset{\pm}{s}_x(\nu).$$

Since μ, ν have compact support so do μ_x, ν_x and since $(\mu * \nu)_x = \mu_x * \nu_x$ (T) implies that

$$(2) \qquad s^{\pm}((\mu * \nu)_x) = s^{\pm}(\mu_x) + s^{\pm}(\nu_x).$$

Because, $\mathcal{S}(\mu * \nu) \subset \mathcal{S}(\mu) + \mathcal{S}(\nu)$

$$(3) \qquad \overset{-}{s}_x(\mu) + \overset{-}{s}_x(\nu) \leqq \overset{-}{s}_x(\mu * \nu) \leqq \overset{+}{s}_x(\mu * \nu) \leqq \overset{+}{s}_x(\mu) + \overset{+}{s}_x(\nu)$$

and hence it suffices to show that strict inequality cannot occur at the extremes of (3). Suppose then that for $c > 0$

$$\overset{+}{s_x}(\mu * \nu) + 2c = \overset{+}{s_x}(\mu) + \overset{+}{s_x}(\nu). \tag{4}$$

If we had simultaneously

$$\overset{+}{s_x}(\mu) < \overset{+}{s}(\mu_x) + c, \quad \overset{+}{s_x}(\nu) < \overset{+}{s}(\nu_x) + c,$$

then upon addition and making use of (2) and (4) we get

$$\overset{+}{s_x}(\mu * \nu) + 2c < \overset{+}{s}((\mu * \nu)_x) + 2c$$

which contradicts (1). Thus either

$$\overset{+}{s_x}(\mu) \geqq \overset{+}{s}(\mu_x) + c \tag{5}$$

or

$$\overset{+}{s_x}(\nu) \geqq \overset{+}{s}(\mu_x) + c \tag{6}$$

holds.

The crucial point is that we may now repeat the argument which leads to (5) or (6) arguing with the measure $\hat{t}\mu$ and $\hat{t}\nu$ instead of μ and ν. Observing that $(\hat{t}\mu) * (\hat{t}\nu) = \hat{t}(\mu * \nu)$ and that $\mathcal{S}(\hat{t}\mu) = \mathcal{S}(\mu)$, and defining

$$M = \left\{\hat{t} \in \hat{G} : \overset{+}{s_x}(\mu) \geqq \overset{+}{s}((\hat{t}\mu)_x) + c\right\},$$
$$N = \left\{\hat{t} \in \hat{G} : \overset{+}{s_x}(\nu) \geqq \overset{+}{s}((\hat{t}\nu)_x) + c\right\},$$

we have that $M \cup N = \hat{G}$. Now let μ_1, ν_1 be the restrictions of μ, ν to $\left\{t \colon x(t) \geqq s_x^+(\mu) - c\right\}$, $\left\{t \colon x(t) \geqq s_x^+(\nu) - c\right\}$. Then M, N are contained in the set of zeros of $\hat{\mu}_1$, $\hat{\nu}_1$ resp. Since these sets are closed, we conclude that either $\hat{\mu}_1$ or $\hat{\nu}_1$ vanishes on an open set. But the lemma then implies that either μ_1 or $\nu_1 = 0$ which contradicts the definition of $s_x^+(\mu)$, $s_x^+(\nu)$. An analogous argument holds for $s_x^-(\mu * \nu)$ and the proof of the theorem is complete.

A version of (T) holds when supports of μ and ν are semibounded, with the obvious conventions $+\infty = +\infty + \infty = +\infty + a$ etc. The same versions hold in general by the above reasoning. In the same way Titchmarsh's theorem on locally integrable functions with support in a strictly convex cone may be extended to this general setting.

Finally, we mention one extension to a nonlocally compact situation, namely a Banach space. Replacing "compact support" with "bounded support," the analogue of the lemma holds and the theorem holds, where clearly $\operatorname{co}(A)$ has now its usual meaning.

The technique used here provides a general method for extending results from $\boldsymbol{R}$ to groups G with connected dual $\hat{G}$. For example, one can easily prove an analogue of Pólya and Plancherel's [5] generalization of a Paley-Wiener theorem in this way, the details are omitted.

References

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