ON SEMICONNECTED MAPPINGS OF TOPOLOGICAL SPACES

JOHN JONES, JR.

1. Introduction. Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ denote topological spaces as in Kelly [1]. A mapping f of $(X, \mathfrak{U})$ into $(Y, \mathfrak{V})$ is said to be connected if and only if it maps connected subsets of $(X, \mathfrak{U})$ into connected subsets of $(Y, \mathfrak{V})$. W. J. Pervin and N. Levine [3] and T. Tanaka [4] recently considered connected mappings of Hausdorff spaces $(X, \mathfrak{U})$ into $(Y, \mathfrak{V})$. A mapping f of $(X, \mathfrak{U})$ into $(Y, \mathfrak{V})$ is semiconnected if $f^{-1}(A)$ is a closed and connected set in $(X, \mathfrak{U})$ whenever A is a closed and connected set in $(Y, \mathfrak{V})$. A mapping f is bi-semiconnected if and only if f and f^{-1} are each semiconnected. Using the definition of G. T. Whyburn [5] a connected T_1-space $(X, \mathfrak{U})$ is said to be semilocally connected (s.l.c.) at $x \in X$ if and only if there exists a local open base at $x \in X$ such that $X \setminus V$ has only a finite number of components, where V is any element of the local open base at x.

Since continuous mappings are special cases of connected mappings it is of interest to know what conditions must be placed upon a given mapping or upon the topological spaces $(X, \mathfrak{U})$, $(Y, \mathfrak{V})$ in order to conclude that a given mapping f is continuous or is a homeomorphism. Examples of connected mappings which are not continuous are given by C. Kuratowski [2] and Pervin and Levine [3].

2. Results.

THEOREM 1. *Let f be a one-to-one onto semiconnected mapping of a topological space $(X, \mathfrak{U})$ to a semilocally-connected topological T_2 space $(Y, \mathfrak{V})$, then f is continuous.*

Let B be an open set in Y, and $f^{-1}(B) = A \subseteq X$. Choose a point $x \in A$ and let $f(x) = y \in B$. Since $(Y, \mathfrak{V})$ is semilocally-connected there exists an open set $B_y \subseteq B$ containing y and $Y \setminus B_y$ consists of a finite number of distinct components. Let these components be designated by $B_1, B_2, B_3, \cdots, B_n$. Then $B_y = Y \setminus \bigcup_{i=1}^{n} B_i$, where each B_i is connected and closed. Let $A_i = f^{-1}(B_i)$, for $i = 1, 2, \cdots, n$. Each A_i is closed and connected since f is semiconnected. Now either x belongs to the closure of some A_j or it does not. Suppose that x belongs to the closure of some A_j, for some $j = 1, 2, \cdots, n$. Now $A_j \cup x$ is closed and connected since f was a semiconnected mapping. Thus $f(A_j \cup x) = B_j$

Received by the editors October 20, 1966 and, in revised form, December 15, 1966.

which is connected. So $y \in B_j$, but this is impossible since B_j is a component of $Y \backslash B_y$ which does not contain y. So x cannot belong to the closure of A_j for any $j=1, 2, 3, \cdots, n$.

There exists an open set $O_i = X \backslash \mathrm{cl}\, A_i$ such that $x \in O_i$, $i=1, 2, 3, \cdots, n$. Now $O_x = \bigcap_{i=1}^n O_i$ is an open set in X containing x. Also $f(A_i) = B_i$, and $f(\mathrm{cl}\, A_i) \supseteq B_i$. So we have $Y \backslash f(\mathrm{cl}\, A_i) \subseteq Y \backslash B_i$. Also

$$f(O_x) = f\left(\bigcap_{i=1}^n O_i\right) = \bigcap_{i=1}^n f(O_i) = \bigcap_{i=1}^n f(X \backslash \mathrm{cl}\, A_i)$$

$$= \bigcap_{i=1}^n [Y \backslash f(\mathrm{cl}\, A_i)] \subseteq \bigcap_{i=1}^n [Y \backslash B_i] = Y \backslash \bigcap_{i=1}^n B_i = B_y.$$

Since f is a one-to-one onto mapping and for any B_y there exists an O_x such that $f(O_x) \subseteq B_y$, f is a continuous mapping of $(X, \mathfrak{U})$ into $(Y, \mathfrak{V})$.

THEOREM 2. *Let* $(X, \mathfrak{U})$, $(Y, \mathfrak{V})$ *be semilocally-connected and* $\wedge T_2$ *topological spaces and* f *be a one-to-one bi-semiconnected mapping of* $(X, \mathfrak{U})$ *onto* $(Y, \mathfrak{V})$, *then* f *is a homeomorphism.*

According to Theorem 1 above f is continuous, and applying the same type argument as used in the proof of Theorem 1, we see that f^{-1} is a one-to-one onto continuous mapping of $(Y, \mathfrak{V})$ to $(X, \mathfrak{U})$. Hence f is a homeomorphism.

The author wishes to acknowledge the helpful comments of the referee.

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GEORGE WASHINGTON UNIVERSITY