NONGENERATORS OF RINGS

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The purpose of this note is to examine the role of nongenerators in the theory of rings, i.e. the elements x of a ring R such that for each subset M of R for which $R=\langle x, M\rangle$, then $\langle M\rangle = R$. The approach used considers a ring as a group with multiple operators and consequently an ideal A generated by a subset S implies that $S\subseteq A$. These results will include those of L. Fuchs [1] and A. Kertesz [2] whenever the ring has unity.

Unless otherwise indicated, the terminology and the necessary known results may be found in N. McCoy's text [3].

Denote the ideal (right ideal) generated by the set M of R by $\langle M\rangle$ ($\langle M\rangle_r$).

DEFINITION. An element $x\in A$ is a *generator* of an ideal (right ideal) A in a ring R provided that there is a subset M of A such that $A=\langle x, M\rangle$ ($A=\langle x, M\rangle_r$) and $\langle M\rangle\subset A$ ($\langle M\rangle_r\subset A$) properly. Otherwise x is called a *nongenerator* of A. (Note that M may be empty.)

The set of nongenerators of an ideal (right ideal) A in a ring R will be denoted by Φ (Φ_r), respectively.

Immediate consequences of the definition are the following:

(i) For an element x of a ring R, $x\in\Phi$ ($x\in\Phi_r$) if and only if $\langle x\rangle\subseteq\Phi$ ($\langle x\rangle_r\subseteq\Phi_r$).

(ii) In a ring R, Φ is an ideal and Φ_r is a right ideal.

Throughout this paper a *maximal ideal* of a ring R will be a proper ideal of R that is not contained in another proper ideal of R. Similarly for maximal right (left) ideals.

(iii) In a ring R, Φ (Φ_r) is the intersection of the maximal ideals (right ideals), if they exist, and is R otherwise.

(iv) For a ring R and homomorphism θ of R, $\Phi\theta\subseteq\Phi(R\theta)$ and $\Phi_r\theta\subseteq\Phi_r(R\theta)$.

(v) For an ideal A of a ring R, $A\subseteq\Phi$ implies that $\Phi(R/A)=\Phi/A$ and $A\subseteq\Phi_r$ implies that $\Phi_r(R/A)=\Phi_r/A$.

(vi) For a ring R, if A is an ideal (right ideal) of R, then $\Phi(A)\subseteq\Phi$ ($\Phi_r(A)\subseteq\Phi_r$).

(vii) In a ring R, $\Phi=(0)$ ($\Phi_r=(0)$) implies that $\Phi(A)=(0)$ ($\Phi_r(A)=(0)$) for each ideal (right ideal) A of R. Such rings will be called *Φ-free* or *Φ_r-free* respectively.

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(viii) In a ring R, $A=\Phi(A)$ $(A=\Phi_r(A))$ for an ideal (right ideal) A of R implies that $A\subseteq\Phi$ $(A\subseteq\Phi_r)$.

(ix) If R is a ring and $R=M_1\oplus\cdots\oplus M_n$, then $\Phi=\Phi(M_1)\oplus\cdots\oplus\Phi(M_n)$ for ideals M_i of R.

(x) In a ring R, if A is a minimal ideal (right ideal) such that $A\not\subseteq\Phi$ $(A\not\subseteq\Phi_r)$, then there exists a maximal ideal (right ideal) B such that $R=A\oplus B$.

(xi) If R is a zero ring $(R^2=(0))$, then $\Phi=\Phi(R^+)$, $\Phi(R^+)$ the Frattini subgroup of the additive group R^+.

In the remaining portion of this note, the Jacobson radical and the upper Baer radical will be denoted by J and N respectively.

THEOREM 1. *In a ring R, $\Phi_r\subseteq J$ and $\Phi\subseteq N$.*

PROOF. If $J\neq R$, then J is the intersection of the modular maximal right ideals of R; and if $N\neq R$, then N is the intersection of the modular maximal ideals.

Note that in Theorem 1 equality may not occur as the ring $\{0, 2; \bmod 4\}$ exemplifies.

THEOREM 2. *In a ring R, $RJ\subseteq\Phi_r$ and $JR\subseteq\Phi_l$, Φ_l denoting the set of nongenerators with respect to left ideals.*

PROOF. Since the result follows if $\Phi_r=R$, consider the case that $\Phi_r\subset R$ properly. Suppose there is an element $x\in R$ such that $yx\notin\Phi_r$ for some element $y\in R$. Then there exists a maximal right ideal M such that $yx\notin M$. M defines a simple right R-module $R/M\cong R^*$, and under the natural R-homomorphism θ of $R\to R^*$, $y\theta\neq 0$ and $(yx)\theta\neq 0$. So $(R^*)R=R^*$, and an element $z\in R$ exists such that $(yxz)\theta=y\theta$. Then note that if xz is r.q.r., there exists an element $b\in R$ such that $xz+b=xzb$. Under θ, $yxz+yb=yxzb$ becomes $(yxz)\theta=-(yb)\theta+(yxzb)\theta=-(yb)\theta+(yb)\theta=0$. So $yxz\in M$ and a contradiction. Therefore xz cannot be r.q.r. In conclusion, if x has the property that $yx\notin\Phi_r$ for some $y\in R$, then $x\notin J$. So for each element $x\in J$, $Rx\subseteq\Phi_r$, i.e. $RJ\subseteq\Phi_r$. Similarly $JR\subseteq\Phi_l$. (*Note*: this proof was suggested by a result of Kertesz [2].)

COROLLARY 2.1. (a) *For a ring R, Φ_r and Φ_l are ideals in R.*

(b) *For a ring R, $J^2\subseteq\Phi_r\cap\Phi_l$.*

(c) $J=(\Phi_r\colon R)=(\Phi_l\colon R)$

(d) *For a ring R, $x\in J$ iff $R\times R\subseteq\Phi_r\cap\Phi_l$.*

Since in general both Φ_r and Φ_l are in J, it follows that in each primitive ring the right ideals and the left ideals are Φ_r- and Φ_l-free respectively. If the ring is a simple nonradical ring, then the ring is

Φ-free. For the simple primitive rings, all three hold. And for a field F, $\Phi(F)=(0)$.

In general $\Phi \not\subseteq J$. For example: let R be the ring of all linear transformations of a vector space V with a denumerable basis. It is known (e.g., see [3]) that R is a primitive ring and $J=(0)$. Since R has unity, $N=R$; and, in fact, the only proper ideal besides (0) is the ideal of elements of finite rank. This ideal is $N=\Phi$. Also note that $\Phi_r=\Phi_l=(0)$.

THEOREM 3. *For a ring R having $R^2=R$, Φ is a semiprime ideal.*

PROOF. Each maximal ideal is prime. If A is an ideal for which $A^2 \subseteq \Phi$, then A^2 is contained in each maximal ideal M. So A is contained in each M. Therefore $A \subseteq \Phi$.

COROLLARY 3.1. *For a ring R having $R^2=R$, the prime radical is contained in Φ.*

COROLLARY 3.2. *For a ring R having $R^2=R$, $J \subseteq \Phi$ iff $J^2 \subseteq \Phi$.*

THEOREM 4. *For a ring R having $R^2=R$ and center Z, $N \cap Z \subseteq \Phi_r$ and $N \cap Z \subseteq \Phi$.*

PROOF. If $A=N \cap Z \not\subseteq \Phi_r$ and M is a maximal right ideal not containing A, then $R=A+M$. This implies that M is a maximal ideal, and $R^2=R$ implies that R/M is a simple commutative nonzero ring. Hence M is modular and $N \subseteq M$ implies that $A \subseteq M$. So $A \subseteq \Phi_r$. Similarly $N \cap Z \subseteq \Phi$.

COROLLARY 4.1. *If R is a commutative ring and $R^2=R$, then $J=\Phi$.*

THEOREM 5. *For a ring R having $R^2=R$, $\Phi_r=\Phi_l=J$.*

PROOF. Consider Φ_r and note that for $R=\Phi_r$ and $\Phi_r \subseteq J$ implies that $J=\Phi_r$. So then consider the case that $\Phi_r \subset R$ properly. By Theorem 2, $J^2 \subseteq \Phi_r$. Form $R/J^2 \cong R^*$ noting that $J^* \cong J(R/J^2)=J/J^2$ and that $\Phi_r^* \cong \Phi_r(R/J^2)=\Phi_r/J^2$. If $x \in J^*$ and $x \notin \Phi_r^*$, there exists a maximal right ideal M^* such that $x \notin M^*$. Under the natural R^*-homomorphism θ of $R^* \to R^*/M^*$, R^* is mapped onto a simple right R^*-module R^*/M^*. Since $x \notin M^*$, then $J^*\theta=R^*/M^*$. But $J^{*2}\theta=(0)$ implies that R^*/M^* is annihilated by R^*, i.e. $(R^*/M^*)R^*=(0)$. This contradicts the hypothesis that $R^2=R$ since, in turn, this implies that $R^{*2}=R^*$ and $(R^*/M^*)R^*=R^*/M^*$. So $J^* \subseteq M^*$. This leads to $J^* \subseteq \Phi_r^*$ and hence $J \subseteq \Phi_r$. So the result follows.

Similarly $\Phi_l=J$.

COROLLARY 5.1 (L. FUCHS [1]). *For a ring with unity, $\Phi_r=\Phi_l=J$.*

THEOREM 6. *If R satisfies the d.c.c. on right ideals, then $\Phi=(0)$ if and only if R is a direct sum of a finite collection of simple ideals.*

PROOF. Consider the intersections of all finite collections of maximal ideals. By the d.c.c. on right ideals, each linear system has a minimal element, say D. If M is a maximal ideal, then $D=D\cap M$. So $D\subseteq\Phi$ and $D=(0)$. As is known, if there exists in a ring a finite number of maximal ideals M_i $(i=1, \cdots, n)$ with zero intersection, then R is isomorphic to the direct sum of some or all the simple rings R/M_i $(i=1, \cdots, n)$. By (ix) each direct summand has $\Phi(R/M_i)=(0)$ since R/M_i is a simple ideal.

Again by (ix) the converse is evident.

THEOREM 7. *If R is a ring with d.c.c. on right ideals, then both Φ_r and Φ_l are contained in Φ.*

PROOF. The theorem is valid whenever $R=\Phi$, so consider the case that $\Phi\subset R$ properly. In particular restrict attention to $R^*=R/M$ for a maximal ideal M. For either $R^{*2}=(0)$ or $R^{*2}=R^*$, $\Phi_r^*=(0)$. Hence under the natural homomorphism θ of $R\rightarrow R^*$, $\Phi_r\theta\subseteq(0)$ implies that $\Phi_r\subseteq M$. So $\Phi_r\subseteq\Phi$ and similarly $\Phi_l\subseteq\Phi$.

THEOREM 8. *If R is a ring with the d.c.c. on right ideals, then $\Phi_r=\Phi_l=\Phi$.*

PROOF. Since Φ_r is an ideal of R form $R^*\cong R/\Phi_r$ having $\Phi_r^*=\Phi_r(R^*)=(0)$, $\Phi^*\cong\Phi/\Phi_r$ and $J^*\cong J/\Phi_r$. If M^* is a maximal right ideal such that $\Phi^*\not\subseteq M^*$, then $R^*=\Phi^*+M^*$. However, since $R^*J^*\subseteq\Phi_r^*=(0)$, then Φ^* is in the annihilator of M^*. This implies that M^* is an ideal of R^* and hence a contradiction to the assumption that $\Phi^*\subseteq M^*$. So $\Phi^*=(0)$, i.e. $\Phi\subseteq\Phi_r$, and the result follows. Similarly, $\Phi_l=\Phi$.

THEOREM 9. *For a ring R with d.c.c. on right ideals and R not a radical ring, then $\Phi=J$ if and only if $R^2=R$.*

PROOF. Suppose $R^2=R$ and there exists a maximal ideal M such that $J\not\subseteq M$. Then under the natural homomorphism θ of $R\rightarrow R/M=R^*$, $J\theta=R^*$. However, since $J^2\subseteq\Phi\subseteq M$ it follows that $R^{*2}=(0)$ and this contradicts $R^2=R$. So $J\subseteq\Phi$. Since $J=N$ and $\Phi\subseteq N$, then $\Phi=J$.

On the other hand, suppose that $J=\Phi\subset R$ properly. Form $R/\Phi\cong R^*$ and note that $J^*\cong J(R/\Phi)=(0)=\Phi^*\cong\Phi(R/\Phi)$. As is known, $J^*=(0)$ implies that $R^{*2}=R^*$. If $R^2\subset R$ properly and $R^2\theta=R^*$ under the natural homomorphism θ of $R\rightarrow R^*$, then $R=\Phi+R^2=R^2$ and a contradiction. So $R^2=R$.

In a radical ring R the condition $\Phi = J$ does not necessarily imply that $R^2 = R$. For example, let R be a zero ring having R^+ a group of type p^∞. Then $\Phi(R) = \Phi(R^+) = R^+$ and $J = R$.

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