A NOTE ON THE CHARACTERISTIC NUMBERS OF LINEAR SYSTEMS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

JOHN ABRAMOWICH

Introduction. The question of stability of solutions of ordinary linear differential equations with variable coefficients is closely related to the determination of their characteristic numbers 1. Of interest in this connection is the relation between the characteristic numbers of the solutions and the generalized characteristic roots of the system. L. Markus [2] has shown that under certain conditions upper and lower bounds for the characteristic numbers can be determined from the generalized characteristic roots. In this paper we show that a similar result holds for a certain class of linear differential-difference equations which includes the class treated in [2]. Namely, we establish a relation between the characteristic numbers of the solutions of the system

$$x_i'(\phi_i(t)) = \sum_{j=1}^{n} p_{ij}(t)x_j(\phi_j(t)); \qquad i = 1, \cdots, n, \ t \geqq 0, \tag{1}$$

and the generalized characteristic roots of the determinant

$$| q_{ij}(t) - 2\delta_{ij} | = 0, \tag{2}$$

where $q_{ij}(t) = p_{ij}(t)\phi_i'(t) + p_{ji}(t)\phi_j'(t)$ and δ_{ij} is the Kronecker symbol ($\delta_{ij} = 0$, $i \neq j$ and $\delta_{ii} = 1$).

We recall some pertinent definitions. The real number μ is said to be the type or Lyapounov number of the real function $f(t)$ $(0 \leqq t < \infty)$ if $\limsup_{t\to\infty} |f(t)| \exp(-\mu' t) = 0$, $\mu' > \mu$, and $\limsup_{t\to\infty} |f(t)| \exp(-\mu'' t) = \infty$, $\mu'' < \mu$. If $|f(t)| \exp(-\mu t)$ is unbounded as $t \to \infty$ for all positive μ, we say that the type number is equal to $+\infty$, if $|f(t)| \exp(-\mu t)$ approaches zero for all negative μ, we say the type number is equal to $-\infty$. It is easily seen that the type number of the function $f(t)$ is $\limsup_{t\to\infty} (1/t) \log |f(t)|$. By the characteristic number of the real valued vector function $x(t)$ $(0 \leqq t < \infty)$ we shall mean the largest of the type numbers of its components.

If $a = (a_1, \cdots, a_n)$ is any real vector, we shall denote $|a|_2 = a_1^2 + \cdots + a_n^2$ and $|a|_1 = |a_1| + \cdots + |a_n|$.

If $E_j = \{\phi_j(t) \mid \phi_j(t) \leqq 0, \ t \geqq 0\}$, $j = 1, \cdots, n$, then $E = \cup E_j$ is called the initial set for the system (1). Assume that $\phi_j(t) \to \infty$, as $t \to \infty$, $j = 1, \cdots, n$, then by a solution of (1) we shall mean any continuous

Received by the editors December 2, 1966.

vector function $x(t)$ defined on $E \cup [0, \infty)$ and identically satisfying (1) for $t \geqq 0$.

The trivial solution of (1) is said to be asymptotically stable if $\lim_{t\to\infty} |x(t)|_1 = 0$ for every solution such that $|x(t)|_1 < \epsilon$, $t \in E$, for some $\epsilon > 0$. Thus the trivial solution of (1) is asymptotically stable if there exists an $\epsilon > 0$ such that every solution satisfying for $t \in E$, $|x(t)|_1 < \epsilon$ has a negative characteristic number. For further details on these questions the reader is referred to the excellent works [**3**] and [**4**].

1. We shall make the following assumptions about the system (1):

(a) The functions $p_{ij}(t)$ are continuous and the functions $\phi_j(t)$ are continuously differentiable for all $t \geqq 0$, $i, j = 1, \cdots, n$.

(b) The functions $\phi_j(t)$ are bounded below and the sets $E_j = \{\phi_j(t) \mid \phi_j(t) \leqq 0,\ t \geqq 0\}$ are nonempty, $j = 1, \cdots, n$.

(c) The functions $\phi_j(t) \to \infty$, $t \to \infty$, and each possesses an inverse monotonically increasing to infinity for $t > t_1 > 0$, which we shall denote by $\phi_j^{-1}(t)$.

Under the above conditions the system (1) possesses continuous solutions [**3**] and the following theorem holds.

THEOREM. *Let $m(t)$ and $M(t)$ be the minimum and maximum roots, respectively, of* (2). *Then the characteristic numbers of the continuous solutions of the system* (1) *are contained in the segment* $[a, b]$ *where*

$$b = \limsup_{t\to\infty} \left(\max_i \frac{1}{t} \int_{t_0}^{\phi_i^{-1}(t)} M(s)ds \right),$$

$$a = \limsup_{t\to\infty} \left(\min_i \frac{1}{t} \int_{t_0}^{\phi_i^{-1}(t)} m(s)ds \right).$$

PROOF. For fixed t let $y_i = x_i(\phi_i(t))$ and let $y = (y_1, \cdots, y_n)$. Let us find the extreme values of the function

$$H(t, y) = \sum_{i,j=1}^{n} p_{ij}(t)\phi_j'(t) y_i y_j$$

under the condition $|y|_2 = 1$. Using the method of Lagrange multipliers, we find the extrema of the function $F(t, y) = H(t, y) - \lambda(t)|y|_2$. We obtain

$$0 = \frac{\partial F}{\partial y_k} = \sum_{j=1}^{n} \{p_{jk}(t)\phi_k'(t) + p_{kj}(t)\phi_j'(t)\} y_j - 2\lambda(t) y_k,$$

for $k = 1, \cdots, n$. Thus the Lagrange multipliers are the roots of (2) and multiplying the above equations by y_k and adding, we obtain $H(t, y)=\lambda(t)$ and hence $m(t)\leqq H(t, y)\leqq M(t)$. Consequently, for all real $(y_1, \cdots, y_n)$ and all $t\geqq 0$, we have the inequalities

$$m(t)\,|\,y\,|_2 \leqq H(t, y) \leqq M(t)\,|\,y\,|_2. \tag{3}$$

For the sake of simplicity of notation let us denote the vector function $x_1(\phi_1(t)), \cdots, x_n(\phi_n(t))$ by $x(\bar{\phi}(t))$.

Let $x(t)$ be any nontrivial solution of (1) such that $|x(\bar{\phi}(t_0))|_2=k^2$ for some fixed $t_0\geqq 0$. Consider the function

$$V(t) = |\,x(\bar{\phi}(t))\,|_2 \exp\left(-\int_{t_0}^{t} M(s)ds\right).$$

Using the differential equations (1) and the inequality (3), we find $dV/dt\leqq 0$ for all $t\geqq t_0$ where $x(\bar{\phi}(t))$ is differentiable. Indeed,

$$\frac{dV}{dt} = \frac{d}{dt}|\,x(\bar{\phi}(t))\,|_2 \exp\left(-2\int_{t_0}^{t} M(s)ds\right)$$
$$-2M(t)\,|\,x(\bar{\phi}(t))\,|_2 \exp\left(-2\int_{t_0}^{t} M(s)ds\right).$$

Using the system (1) we find that $(d/dt)|x(\bar{\phi}(t))|_2=2H(t, x(\bar{\phi}(t)))$, except possibly at those values of t where the left-hand side does not exist, and the result follows from (3). Since $x(\bar{\phi}(t))$ is continuous, we have for all $t\geqq t_0$

$$|\,x(\bar{\phi}(t))\,|_2 \leqq k^2 \exp\left(2\int_{t_0}^{t} M(s)ds\right).$$

In a similar way, using the function

$$V(t) = |\,x(\bar{\phi}(t))\,|_2 \exp\left(-2\int_{t_0}^{t} m(s)ds\right),$$

we obtain

$$|\,x(\bar{\phi}(t))\,|_2 \geqq k^2 \exp\left(2\int_{t_0}^{t} m(s)ds\right).$$

Therefore, for all $t\geqq t_0$,

$$k^2 \exp\left(2\int_{t_0}^{t} m(s)ds\right) \leqq |\,x(\bar{\phi}(t))\,|_2 \leqq k^2 \exp\left(2\int_{t_0}^{t} M(s)ds\right). \tag{4}$$

Since $x_i^2(\phi_i(t))\leqq k_1^2 \exp(2\int_{t_0}^{t}M(s)ds)$ implies

$$x_i^2(t) \leqq k_2^2 \exp\left(2\int_t^{\phi_i^{-1}(t)} M(s)ds\right) \leqq k_2^2 \exp\left(\max_i 2\int_t^{\phi_i^{-1}(t)} M(s)ds\right),$$

we have

$$|\,\bar{x}(t)\,|_2 \leqq k' \exp\left(\max_i 2\int_{t_0}^{\phi_i^{-1}(t)} M(s)ds\right)$$

for some positive constant k' and by assumption (c) for all $t \geqq t_0$. In a similar way, we show that

$$|\,x(t)\,|_2 \geqq k'' \exp\left(\min 2\int_{t_0}^{\phi_i^{-1}(t)} m(s)ds\right)$$

for some positive constant k'' and $t \geqq t_0$. Thus

$$\begin{aligned} k'' \exp\left(\min_i \int_t^{\phi_i^{-1}(t)} m(s)ds\right) &\leqq |\,x(t)\,|_2 \\ &\leqq k' \exp\left(\max_i 2\int_{t_0}^{\phi_i^{-1}(t)} M(s)ds\right). \end{aligned} \tag{5}$$

Since the type number of $f^2(t)$ is twice that of $f(t)$, the characteristic number of $x(t)$ is equal to half the type number of $|x(t)|_2$ and the theorem follows from the above inequalities.

COROLLARY 1. *If*

$$\lim_{t\to\infty} \frac{1}{t}\left(\max_i \int_{t_0}^{\phi_i^{-1}(t)} M(s)ds\right) = -c, \qquad c > 0,$$

then the trivial solution of (1) *is asymptotically stable.*

COROLLARY 2. *If*

$$\limsup_{t\to\infty} \frac{1}{t}\left(\min_i \int_{t_0}^{\phi_i^{-1}(t)} m(s)ds\right) = c, \qquad c > 0,$$

then the trivial solution of (1) *is unstable.*

REMARK. Assumption (c) was required to insure that the inequalities (4) and (5) hold for all $t \geqq t_0$. These could hold on some finite interval without this assumption.

As an example let us take the second order system

$$\begin{aligned} x'(t) &= (\sin t - \cos^2 t)x(t) + p_{12}(t)y(\tfrac{1}{2}t), \\ y'(\tfrac{1}{2}t) &= p_{21}(t)x(t) - 4(\sin t + 2\cos^2 t)y(\tfrac{1}{2}t), \end{aligned}$$

where $p_{12}(t)+\frac{1}{2}p_{21}(t)=0$.

We find $M(t)=\sin t-\cos^2 t$ and

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^{2t}(\sin s-\cos^2 s)ds=-1$$

and therefore by Corollary 1 the trivial solution is asymptotically stable.

References

1. A. M. Lyapounov, *Problème générale de la stabilité du mouvement*, Princeton Univ. Press, Princeton, N. J., 1947.

2. L. Markus, *Continuous matrices and the stability of differential systems*, Math. Z. **62** (1955), 310–319.

3. L. E. Elsgolts, *Introduction to the theory of differential equations with retarded arguments*, Moscow, 1964. (Russian)

4. R. Bellman and K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.

Wayne State University