SPECIAL DIVISORS ON COMPACT RIEMANN SURFACES[1]

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Introduction. In a previous paper [1] the author obtained some results concerning the distribution of special divisors on compact Riemann surfaces of genus g. The reader is referred to [1] for definitions and notation.

It was shown in [1, p. 886] that the product structure with $S_r \times S_r$ and $T^g(S)$ as factors may be endowed with a structure of complex analytic manifold in such a way that the resulting space, $W_{r,r}$, is an analytic fibre space over the base manifold $T^g(S)$ with fibre $S_r \times S_r$ over $S(T) \in T^g(S)$. We then proved [1, Theorem 2] that if ζ, ω are completely distinct equivalent special divisors of degree r on S_0, then considering the triple $(\zeta, \omega, S_0(T))$ as a point in $W_{r,r}$, there is a g codimensional submanifold of $W_{r,r}$ containing the point $(\zeta, \omega, S_0(T))$ (each point of which has projections onto pairs of equivalent, special divisors on the surface $S_0(T)$, the base point under the fibre) which projects onto a λ codimensional submanifold of $T^g(S)$. Bounds were obtained for λ, and from these bounds it followed that a special divisor of degree $r < (g+1)/2$ is always special in the sense of moduli. Finally we showed that if g is odd, a special divisor of degree $(g+1)/2$ is also special in the sense of moduli. Hence our results for special divisors were [1, Theorem 5] that if g is even (odd), a special divisor of degree less than $(g+2)/2$ $((g+3)/2)$ is special in the sense of moduli. As a particular example of the method, we computed the dimension of the sublocus of $T^g(S)$ possessing Weierstrass points whose Weierstrass sequences begin with a fixed $r < g$.

It is the purpose of this note to indicate that the techniques used in [1] can be employed to yield a general theorem from which the results of [1] emerge as corollaries. Furthermore, in our present treatment, we shall obtain directly that a special divisor of degree less than $(g+2)/2$ is special in the sense of moduli, eliminating the necessity of Theorem 4 in [1].

1. Suppose f is a meromorphic function with r zeros and r poles on S a compact Riemann surface of genus g. Then, by Abel's theorem, S possesses a pair of integral, equivalent, completely distinct, special divisors of degree r, and f projects S onto an r-sheeted branched cover

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of the Riemann sphere with $2r+2g-2$ branch points. Conversely, if S permits a representation as an r-sheeted cover of the sphere, S possesses a pair of equivalent, integral, completely distinct, special divisors of degree r. Hence we see that the existence of a pair of integral, equivalent, completely distinct, special divisors on S is equivalent to the existence of an r-sheeted concrete representation of the surface S. With this in mind, the results of [1] mentioned in the introduction can be restated in the following way: for g even (odd) the property of permitting an r-sheeted representation is special in the sense of moduli if $r<(g+2)/2$ $((g+3)/2)$.

Having stated the results in this form, the following question naturally presents itself. Let S be a compact Riemann surface of genus g which permits an r-sheeted concrete representation with an $(s-1)$th order branch point $2\leqq s\leqq r$. When is such a representation special in the sense of moduli? Before we answer this question, we observe that the desired property is equivalent to the demand that S possess a pair of integral, equivalent, special divisors $\zeta=P^sP_1\cdots P_{r-s}$, $\omega=Q_1\cdots Q_r$. This follows by virtue of the fact that we can always arrange that the branch point of order $s-1$ lie over the origin or ∞.

We recall that an integral divisor ζ of degree $r<g$ is said to be special if $i(\zeta)=g-r+1+m$, $m\geqq 0$.

THEOREM 1. *The dimension d of the locus of $T^g(S)$ whose underlying surfaces permit r-sheeted representations with an $(s-1)$th order branch point satisfies the inequalities*

$$r+2g-4-m\leqq d\leqq \min(V-m-(s+1),3g-3)$$

where $V=2r+2g-2$ and m is computed from the index of specialty of the divisor.

PROOF. As we have already indicated, the desired property is equivalent to the existence on S of integral, equivalent, completely distinct, special divisors $\zeta=P^sP_1\cdots P_{r-s}$, $\omega=Q_1\cdots Q_r$.

By Abel's theorem

$$u_i(\zeta)-u_i(\omega)-m_i-\sum_{j=1}^{g} n_j\Pi_{ij}=0,\qquad i=1,\cdots g,$$

where m_i, n_j are integers. These equations are of course the same as

$$su_i(P)+u_i(P_1)+\cdots+u_i(P_{r-s})-u_i(Q_1)-\cdots-u_i(Q_r)-m_i$$
$$-\sum_{j=1}^{g} n_j\Pi_{ij}=0.$$

We now view these equations as ranging over $W_{r-s+1,r}$, and just as in the proof of Theorem 2 in [1], define therein a g codimensional submanifold, each point of which has projections onto pairs of equivalent, integral, special divisors of the specified form which projects onto a λ codimensional submanifold of $T^g(S)$. The same sort of estimates which yielded Theorem 3 in [1] give here that

$$g - 2r + m + s \leqq \lambda \leqq g - r + 1 + m$$

and we therefore immediately obtain the desired inequalities on d.

We now restrict ourself to the case $m=0$. If $m>0$, then S permits a concrete representation with fewer than r sheets. If $m=0$ the inequalities on d are

$$r + 2g - 4 \leqq d \leqq \min(V - (s + 1), 3g - 3).$$

COROLLARY *Let S be a compact Riemann surface of genus g. The property of S permitting an r-sheeted concrete representation with an $(s-1)$th order branch point is special in the sense of moduli if $r<(g+s)/2$.*

PROOF. By definition, the property is special if $d<3g-3$. $d<3g-3$ if $r<(g+s)/2$.

2. In [1] we considered arbitrary, integral, special divisors of degree $r<g$. These special divisors of course give rise to meromorphic functions which project the Riemann surface onto a concrete r-sheeted cover of the sphere with $2r+2g-2$ branch points. The point is, and this is what we omitted mentioning in [1], that even if the branch points are all simple we can still assume that the special divisor is of the form $P^2P_1 \cdots P_{r-2}$. Hence, we can apply our theorem with $s=2$ and obtain the result that a special divisor of degree $<(g+2)/2$ is always special in the sense of moduli, and we do not have to appeal to Theorem 4 of [1].

In the case of a Weierstrass point whose Weierstrass sequence begins with a fixed $r<g$, we have $s=r$ and hence the property is special if $r<(g+r)/2$ or if $r<g$. Furthermore, in this case, the inequalities satisfied by d are

$$r + 2g - 4 \leqq d \leqq r + 2g - 3.$$

Returning for a moment to the case $s=2$ and $m=0$, we see that in this case the inequalities on d are

$$r + 2g - 4 \leqq d \leqq \min(V - 3, 3g - 3).$$

Hence, we see that if there are few enough branch points, we obtain the result of Riemann that after normalizing 3 branch points the number that remain are an upper bound for the dimension of the space of moduli. In connection with these remarks the reader is referred to [2] for a more complete discussion.

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References

1. H. M. Farkas, *Special divisors and analytic subloci of Teichmueller space*, Amer. J. Math. **88** (1966), 881–901.

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