PR-FACTORIZATIONS OF FAMILIES OF LIGHT INTERIOR FUNCTIONS[1]

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Introduction. A familiar technique used in dealing with problems which fall into a certain class is to find those problems which can be converted into some canonical form which is more easily analyzed. In this paper I shall be considering families of light interior functions defined in a domain $\mathfrak{D}$ in E^2 and the canonical forms will be families of *Bers* functions.

1. **Preliminary concepts and definitions.** If f is a C' function, the *Jacobian* matrix of f will be denoted by $J(f)$ and the determinant of $J(f)$ will be denoted by $|J(f)|$.

DEFINITION 1.1. A C' function f will be said to be *pseudo-regular* in $\mathfrak{D}$ if (i) $|J(f)| \geqq 0$, (ii) $|J(f)| = 0$ if and only if $J(f)$ is the zero matrix, and (iii) the critical points of f are countable and have no limit point in $\mathfrak{D}$. A pseudo-regular function is locally quasiconformal except in a neighborhood of a critical point.

DEFINITION 1.2. A collection $\mathfrak{W}$ of light interior functions defined in $\mathfrak{D}$ will be called a *real linear family* if for f_1 and f_2 in $\mathfrak{W}$, $c_1f_1+c_2f_2$ is in $\mathfrak{W}$ for all real c_1 and c_2. $\mathfrak{W}$ will be said to be *nontrivial* if it contains at least two linearly independent elements.

DEFINITION 1.3. A real linear family $\mathfrak{W}$ will be called a *Bers* family if the elements of $\mathfrak{W}$ are solutions of a *Bers system* $U_x=\tau V_x+\sigma V_y$, $-U_y=\sigma V_x-\tau V_y$, where σ and τ are Hölder-continuous real-valued functions and $\sigma>0$. Elements of a *Bers* family will be called *Bers* functions. (Bers [1] calls these functions "pseudoanalytic functions of the second kind".)

DEFINITION 1.4. A real linear family $\mathfrak{W}$ of light interior functions defined in $\mathfrak{D}$ will be said to have a PR-factorization if there exists a homeomorphism h defined in $\mathfrak{D}$ and a *Bers* family $\tilde{\mathfrak{W}}$ in $h(\mathfrak{D})$ such that if f is in $\mathfrak{W}$, there exists $\tilde{f}$ in $\tilde{\mathfrak{W}}$ for which $f=\tilde{f}\circ h$. We denote this PR-factorization by $[h, \tilde{\mathfrak{W}}]$.

In an earlier paper [3], I showed that every nontrivial real linear family of pseudo-regular functions has a PR-factorization with h a *Beltrami* function. Furthermore, this factorization is not unique and if h_1 and h_2 are *Beltrami* functions satisfying the same *Beltrami* system

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in $\mathfrak{D}$ and corresponding to two distinct PR-factorizations of $\mathcal{W}$, $h_1 \circ h_2^{-1}$ is a conformal mapping of $h_2(\mathfrak{D})$ onto $h_1(\mathfrak{D})$.

2. **Some theorems on PR-factorization.** If $\mathcal{W}$ is a *Bers* family, we may assume that $\mathcal{W}$ contains at least one homeomorphism since every *Bers* system has homeomorphic solutions. It follows that if $\mathcal{W}$ is a maximal nontrivial real linear family which has a PR-factorization, $\mathcal{W}$ contains at least one homeomorphism. Finally, if $\mathcal{W}$ is a *Bers* family, $\mathcal{W}$ always has a PR-factorization of the type $[h, \tilde{\mathcal{W}}]$ where h is conformal.

THEOREM 2.1. *Let $\mathcal{W}$ be a nontrivial real linear family. If $\mathcal{W}$ has a PR-factorization, this PR-factorization is not unique.*

PROOF. Let $[h, \tilde{\mathcal{W}}]$ be a PR-factorization of $\mathcal{W}$. Since $\tilde{\mathcal{W}}$ is a *Bers* family, $\tilde{\mathcal{W}}$ has a PR-factorization $[g, \hat{\mathcal{W}}]$ where g is conformal. If we define $\hat{h} = g \circ h$, $[\hat{h}, \hat{\mathcal{W}}]$ is a PR-factorization of $\mathcal{W}$.

If, in the preceding theorem, we let $h_1 = h$ and $h_2 = h \circ g$, we see that $h_1 \circ h_2^{-1} = g^{-1}$ and $h_2 \circ h_1^{-1} = g$ are conformal. In the general case, $h_1 \circ h_2^{-1}$ will not necessarily be conformal.

THEOREM 2.2. *Let $[h_1, \mathcal{W}_1]$ and $[h_2, \mathcal{W}_2]$ be PR-factorizations of a nontrivial real linear family $\mathcal{W}$ and let τ_i and σ_i, $i = 1, 2$, be the coefficients of the associated Bers systems. Then $h_1 \circ h_2^{-1}$ is pseudo-regular and if $h_1 \circ h_2^{-1}$ is conformal, $\tau_2 = \tau_1 \circ (h_1 \circ h_2^{-1})$ and $\sigma_2 = \sigma_1 \circ (h_1 \circ h_2^{-1})$.*

PROOF. Let f_1 be a homeomorphic element of $\mathcal{W}_1$ and let f_2 be the element of $\mathcal{W}_2$ such that $f_2 = f_1 \circ (h_1 \circ h_2^{-1})$. f_1 and f_2 are pseudo-regular on their respective domains. Since the composition of pseudo-regular functions is a pseudo-regular function and the inverse of a homeomorphic pseudo-regular function is pseudo-regular, $h_1 \circ h_2^{-1} = f_1^{-1} \circ f_2$ is pseudo-regular. Now let $h_1 \circ h_2^{-1} = p + iq$, let $f_1 = u + iv$, and let $f_2 = r + is$. A simple computation shows that

$$\tau_1(x, y) = \tau_2(p(x, y), q(x, y)) + A\sigma_2(p(x, y), q(x, y)) \tag{2.1}$$

and

$$\sigma_1(x, y) = B\sigma_2(p(x, y), q(x, y)) \tag{2.2}$$

where

$$A = \frac{s_p s_q(p_x^2 + p_y^2 - q_x^2 - q_y^2) - (s_p^2 - s_q^2)(p_x q_x + p_y q_y)}{s_p^2(p_x^2 + p_y^2) + 2s_p s_q(p_x q_x + p_y q_y) + s_q^2(q_x^2 + q_y^2)} \tag{2.3}$$

and

$$B = \frac{(s_p^2 + s_q^2)(p_x q_y - p_y q_x)}{s_p^2(p_x^2 + p_y^2) + 2s_p s_q(p_x q_x + p_y q_y) + s_q^2(q_x^2 + q_y^2)}. \tag{2.4}$$

If $h_1 \circ h_2^{-1}$ is conformal, $A=0$ and $B=1$.

Since pseudo-regular functions have many properties in common with analytic functions, one might suppose that the theorem on removable singularities could be extended to pseudo-regular functions. This is not true. One can exhibit functions which are homeomorphisms of E^2 into E^2, have partial derivatives at each point in E^2, and are pseudo-regular in the punctured plane but which are not pseudo-regular at the origin. It is apparent from the preceding theorem and from earlier discussions that if $\mathcal{W}$ has a PR-factorization and if $\mathcal{W}$ contains a homeomorphic pseudo-regular element, then every element of $\mathcal{W}$ is pseudo-regular and h may be taken to be a *Beltrami* function.

THEOREM 2.3. *Let $\mathcal{W}$ be a nontrivial real linear family of light interior functions defined in $\mathfrak{D}$. If $\mathcal{W}$ has a PR-factorization, either there exists a subset E of $\mathfrak{D}$ having no limit point in $\mathfrak{D}$ such that every element of $\mathcal{W}$ is pseudo-regular in $\mathfrak{D}-E$ or $\mathcal{W}$ contains no pseudo-regular elements.*

PROOF. Let $[h, \tilde{\mathcal{W}}]$ be a PR-factorization of $\mathcal{W}$. If f is a pseudo-regular element of $\mathcal{W}$, h is also pseudo-regular except possibly at the critical points of f. Since every element of $\mathcal{W}$ may be represented as the composition of h with a *Bers* function, every element of $\mathcal{W}$ is pseudo-regular in $\mathfrak{D}-E$ where E is the set of critical points of f. Finally, since f is pseudo-regular in $\mathfrak{D}$, E has no limit point in $\mathfrak{D}$.

COROLLARY 2.3. *Let $\mathcal{W}$ be a nontrivial real linear family of light interior functions defined in $\mathfrak{D}$. Let g be a pseudo-regular element of $\mathcal{W}$, and let E be the critical points of g. If $\mathcal{W}$ contains a function which is not pseudo-regular in $\mathfrak{D}-E$, $\mathcal{W}$ has no PR-factorization.*

PROOF. Trivial.

A family satisfying the hypotheses of this corollary will be exhibited in the next section.

3. A nontrivial family which has no PR-factorization. If $f(x, y) = -y+ix$ and $g(x, y)=x^3+iy^3$, f and g are C' homeomorphisms (f is analytic) and it is easy to verify that $\alpha f+\beta g$ is a C' light interior mapping for α and β arbitrary real numbers. Restricting ourselves for the moment to the open first quadrant, f and g determine the first order system

$$U_x = (x^2/y^2)V_y, \qquad -U_y = V_x. \tag{3.1}$$

Obviously, f and g are solutions of (3.1). By quite elementary methods, one may obtain a five-parameter family of solutions of (3.1) given by

$$\begin{aligned} U &= x^{3/2}y^{1/2}[C_1I_{1/4}(\alpha y^2) - C_2I_{-1/4}(\alpha y^2)] \\ &\quad\cdot[C_3J_{-3/4}(\alpha x^2) - C_4J_{3/4}(\alpha x^2)], \end{aligned} \tag{3.2}$$

$$\begin{aligned} V &= x^{1/2}y^{3/2}[C_1I_{-3/4}(\alpha y^2) + C_2I_{3/4}(\alpha y^2)] \\ &\quad\cdot[C_3J_{1/4}(\alpha x^2) + C_4J_{-1/4}(\alpha x^2)] \end{aligned} \tag{3.3}$$

where C_1, C_2, C_3, and C_4 are arbitrary real numbers, α is any positive real number, J_p is the Bessel function of order p, and I_p is the modified Bessel function of the first kind of order p. Note that U, V, and their partial derivatives have at most removable discontinuities at $x=0$ and $y=0$ and are continuous for all other values of x and y. Furthermore,

$$\begin{aligned} U_xV_y - U_yV_x &= V_x^2 + \frac{x^2}{y^2}V_y^2 \\ &= 4\alpha^2x^3y^3[C_1I_{-3/4}(\alpha y^2) + C_2I_{3/4}(\alpha y^2)]^2 \\ &\quad\cdot[C_3J_{-3/4}(\alpha x^2) - C_4J_{3/4}(\alpha x^2)]^2 \\ &\quad+4\alpha^2xy[-C_1I_{1/4}(\alpha y^2) + C_2I_{-1/4}(\alpha y^2)]^2 \\ &\quad\cdot[C_3J_{1/4}(\alpha x^2) + C_4J_{-1/4}(\alpha x^2)]^2 \end{aligned}$$

is nonnegative and bounded for all finite values of x and y. If $\mathfrak{W}$ consists of f, g, and functions of the form $u+iv$ where the pair (u, v) satisfy (3.1), it contains a pseudo-regular function and functions which are not pseudo-regular.

References

1. L. Bers, *Theory of pseudo-analytic functions*, New York Univ., 1953.

2. W. V. Caldwell, *Maximal vector spaces of light interior functions*, J. Math. Mech. **12** (1963), 411–428.

3. ———, *Some relationships between Bers and Beltrami systems and linear elliptic systems of partial differential equations*, Canad. J. Math. **17** (1965), 627–642.

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