A GENERALIZATION OF A THEOREM OF HANS LEWY

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Let ρ be a real-valued C^∞ function defined in a neighborhood of the origin 0 in C^n, such that $\rho(0)=0$, $d\rho(0)\neq 0$. Then, near zero, $M=\{z;\ \rho(z)=0\}$ is a real submanifold of C^n of dimension $2n-1$. If $\partial\bar\partial\rho(0)\neq 0$, then M has a holomorphic hull which contains an open set. We shall prove an L^2 version of this fact. Let $\bar\partial_b$ represent the tangential Cauchy-Riemann operator on M introduced by Kohn [1]. By L^2 on M, we mean the space of functions which are square integrable with respect to surface area.

THEOREM. *There is a neighborhood N of 0 such that if $D=\{z\in N;\ \rho(z)<0\}$ and f is an L^2 function on $N\cap M$, the following are equivalent:*

(i) *f is a weak solution of the equation $\bar\partial_b f=0$,*

(ii) *$\int_M f\bar\partial\alpha=0$ for all $(n,\ n-2)$ forms α whose support intersects $N\cap M$ in a compact set,*

(iii) *f is the boundary value of a function holomorphic in D,*

(iv) *f is locally L^2 approximable by functions holomorphic in a neighborhood of $N\cap M$.*

The only nontrivial part of this theorem is (ii) implies (iii) and (iv); this was proven in [2] by Hans Lewy for f a C^1 function ($n=2$, but that does not matter). The proof here is an adaptation of his argument. We need to use the following verifiable lemmas.

LEMMA 1. *Let f be a square integrable function defined in a domain in C^n. f is holomorphic if and only if $\int f\bar\partial\alpha=0$ for all compactly supported $(n,\ n-1)$ forms α.*

LEMMA 2. *Let X be a compact Hausdorff space, μ a finite Baire measure, $\Delta=\{z\in C;\ |z|<1\}$, $\Gamma=\{z\in C;\ |z|=1\}$. Let $f\colon X\to L^2(\Gamma)$ be square integrable; $\int\|f(x)\|^2d\mu<\infty$ and suppose also*

$$\int_\Gamma f(x)(\theta)e^{in\theta}d\theta = 0 \quad \text{for } n > 0.$$

Let $\hat f(x, z)$ for $z\in\Delta$ be the Cauchy integral of $f(x)$. Then $\hat f$ has the bound-

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ary values f. More precisely, let $\phi: X\times\Gamma\times[0, 1]\to\bar{\Delta}$ *be a continuous map with these properties*:

(i) $\phi(x, \theta, t)=\phi(x, t)e^{i\theta}$, $\phi(x, 0)=1$,

(ii) $\phi(X\times\Gamma\times(0, 1))\subset\Delta$.

Then

$$\lim_{\delta\to 0}\int_{X\times\Gamma}|\hat{f}(x, \phi(x, \theta, \delta)) - f(x, \theta)|^2 d\theta d\mu = 0.$$

Now, we return to M. Because $\partial\bar{\partial}\rho(0)\neq 0$ and $d\rho(0)\neq 0$ we may choose complex coordinates $(z, w, w_1, \cdots, w_{n-2})$ near zero so that M is given by

$$0 = \rho(z) = \operatorname{Re} w + z\bar{z} + Q(\zeta, \zeta) + O(2),$$

where ζ is the multivariable $(w_1, \cdots, w_{n-2})$, Q is a quadratic form, and $O(2)$ consists of terms of higher order at 0. Let $\pi: C^n\to C^{n-1}$, $\pi(z, w, \zeta)=(w, \zeta)$. It is shown in [3], that the mapping π has the following structure. There is a ball N, center at 0 such that $\pi(M\cap N)$ is the closure (in $\pi(N)$) of a domain D_0. For $(w, \zeta)\in D_0$, $\Gamma_{(w,\zeta)}=\pi^{-1}(w, \zeta)\cap M$ is a simple closed curve in the z-plane bounding the domain $\Delta_{(w,\zeta)}$. As $(w, \zeta)\to bD_0$, $\Gamma_{(w,\zeta)}\to$point. Let $D=\{(z, w, \zeta); (w, \zeta)\in D_0, z\in\Delta_{(w,\zeta)}\}$. With the situation so given we prove

LEMMA 3. *Let* $f\in L^2$ *on* M *with the property* (ii) *of the theorem. Then*

$$\hat{f}(z, w, \zeta) = \frac{1}{2\pi i}\int_{\Gamma_{(w,\zeta)}}\frac{f(\eta, w, \zeta)d\eta}{\eta - z}$$

is holomorphic in D. *If* B *is a closed ball contained in* N, $f|B$ *is* L^2-*approximable by translates of* $\hat{f}$ *which are holomorphic in a neighborhood of* $B\cap M$.

PROOF. First of all, $\hat{f}$ is clearly locally L^2 in D. We use Lemma 1 to verify that $\hat{f}$ is holomorphic. Let β be a compactly supported (in D), $(n, n-1)$ form. Let $dV'=dw\wedge dw_1\wedge\cdots\wedge d\bar{w}_n\wedge d\bar{w}\wedge\cdots\wedge d\bar{w}_n$.

(i) $\beta=hdz\wedge dV'$. Then

$$\int\hat{f}\bar{\partial}\beta = \int_{D_0}\left\{\int_{\Delta_{(w,\zeta)}}\left[\frac{1}{2\pi i}\int_{\Gamma_{(w,\zeta)}}\frac{\partial h}{\partial\bar{z}}\frac{f(\eta, w, \zeta)d\eta}{\eta - z}\right]d\bar{z}\wedge dz\right\}dV'$$

$$= \frac{1}{2\pi i}\int_{D_0}\left\{\int_{\Gamma_{(w,\zeta)}}f(\eta, w, \zeta)\left[\int_{\Delta_{(w,\zeta)}}\frac{1}{\eta - z}\frac{\partial h}{\partial\bar{z}}d\bar{z}\wedge dz\right]d\eta\right\}dV'.$$

Since η is outside the support of h, by Lemma 1 for $n=1$, the innermost integral is always zero. Thus $\int\hat{f}\bar{\partial}\beta=0$ in this case.

(ii) $\beta=d\bar{z}\wedge dz\wedge\gamma$, where γ is compactly supported $(n-1, n-2)$

with no $d\bar{z}$ or dz term. Then $\bar\partial\beta = d\bar{z}\wedge dz\wedge\bar\partial'\gamma$ where $\bar\partial'\gamma$ is taken as if γ were considered as an $(n-1, n-2)$ form in the (w, ζ)-space, with coefficients varying in z.

Thus

$$\int \hat{f}\bar\partial\beta = \frac{\pm 1}{2\pi i}\int_{D_0}\left\{\int_{\Delta(w,\zeta)}\left[\int_{\Gamma(w,\zeta)} \frac{f(\eta, w, \zeta)d\eta}{\eta - z}\wedge\bar\partial\gamma\right]d\bar{z}\wedge dz\right\}$$
$$= \frac{\pm 1}{2\pi i}\int_{D_0}\left\{\int_{\Gamma(w,\zeta)} f(\eta, w, \zeta)\left[\int_{\Delta(w,\zeta)}\frac{\bar\partial'\gamma}{\eta - z}d\bar{z}\wedge dz\right]d\eta\right\}.$$

Let

$$\alpha(\eta, w, \zeta) = \left(\int_{\Delta(w,\zeta)}\frac{\gamma}{\eta - z}d\bar{z}\wedge dz\right)\wedge d\eta.$$

α is a $C^\infty(n, n-2)$ form defined in a neighborhood of M whose support intersects M in a compact set (since γ vanishes in a neighborhood of M). Further, computing $\bar\partial\alpha$, we find $\int\hat{f}\bar\partial\beta = \int_M f\bar\partial\alpha = 0$ by hypothesis.

Now since any compactly supported $(n, n-1)$ form on D is a sum of forms of type (i) and (ii), we have $\int\hat{f}\bar\partial\beta = 0$ for all such forms, so by Lemma 1, $\hat{f}$ is holomorphic.

Now, in order to apply Lemma 2, we must verify that, for fixed (w, ζ), $\hat{f}(z, w, \zeta)$ has the boundary value f. That is

$$(*)\qquad \int_{\Gamma(w,\zeta)} f(z, w, \zeta)z^n dz = 0 \quad \text{for } n \geqq 0.$$

Let

$$F(w, \zeta) = \int_{\Gamma(w,\zeta)} f(z, w, \zeta)z^n dz \quad w \in D_0,$$
$$= 0 \quad w \notin D_0.$$

We show that F is holomorphic in $\pi(N)$. First, if β is a $C^\infty(n-1, n-2)$ form, compactly supported in D_0,

$$\int F\bar\partial\beta = \int_{D_0}\int_{\Gamma(w,\zeta)} f(z, w)z^n dz\wedge\bar\partial\beta = \int_M f\bar\partial(z^n dz\wedge\beta) = 0.$$

Let β now be any $(n-1, n-2)$ form compactly supported in $\pi(N)$. Choose real C^∞ coordinates in $\pi(N)$, $x_1, \cdots, x_{2n-2}$ so that $bD_0 = \{x_1 = 0\}$, $D_0 = \{x_1 > 0\}$. (We may have to do this locally, but after applying a partition of unity to β this is the general case.) Reducing to the plane $x_2, \cdots, x_{2n-2} = \text{constant}$, $\arg z_1 = \text{constant}$, we see that M intersects this plane in a curve $x_1 = A|z_1|^2 + \cdots$, where A depends differentiably on the other constants and is bounded away from zero. Thus the length of Γ_w is of the order of $2\pi\sqrt{x_1}$.

Now let $\rho(x_1)$ be a C^∞ function such that

$$\begin{aligned} \rho &\equiv 0 \quad \text{when } x_1 \geqq \epsilon, \\ \rho &\equiv 1 \quad \text{when } x_1 \leqq 0, \end{aligned} \qquad | d\rho/dx_1 | \leqq 2/\epsilon.$$

Now $\int_{D_0} F\bar\partial\beta = \int_{D_0} F\bar\partial(\rho\beta)$, since $\int F\bar\partial(1-\rho)\beta = 0$, as above. Now $\bar\partial\rho\beta = \bar\partial\rho \wedge \beta + \rho\bar\partial\beta$,

$$\begin{aligned} \left| \int F\bar\partial\rho \wedge \beta \right| &= \left| \int_{D_0} \int_{\Gamma_{(w,\zeta)}} f(z, w, \zeta) dz \wedge \bar\partial\rho \wedge \beta \right| \\ &= \left| \int_{D_0} \int_{\Gamma_{(w,\zeta)}} \left\{ f(z, x) z^n \frac{d\rho}{dx_1} c_\beta(x) \right\} dz \wedge dV \right| \end{aligned}$$

where c_β is a function depending only on x, and dV is the element of volume in D_0. Using Schwarz's inequality,

$$\left| \int F\bar\partial\rho \wedge \beta \right| \leqq K_n \|f\| \left(\int_{D_0} \left| \frac{\partial\rho}{\partial x_1} c_\beta(x) \right|^2 \int_{\Gamma_{(w,\zeta)}} d|z| \right)^{1/2}.$$

But $\int_{\Gamma_w} d|z| \sim 2\pi\sqrt{x_1}$, thus

$$\left| \int F\bar\partial\rho \wedge \beta \right| \leqq K_n' K_\beta \|f\| \epsilon^{-1} \int_{0 \leq x_1 \leq \epsilon} \sqrt{x_1} dx_1 \leqq \|f\| \sqrt{\epsilon}.$$

Now $\int F\rho\bar\partial\beta$ is even better, so we find, letting $\epsilon \to 0$ that $\int_{D_0} F\bar\partial\beta = 0$. Thus F is holomorphic in $\pi(N)$, and since it is identically zero in an open set, it is identically zero, and (*) is verified.

Now, fix a Riemann map $R_{(w,\zeta)}$ of $\Delta_{(w,\zeta)}$ onto $\{|z| < 1\}$, differentiable at the boundary, and varying differentiably in (w, ζ). Define $\psi: \bar D \to M$, $\psi(z, w, \zeta) =$ the point on $\Gamma_{(w,\zeta)}$ with the same Riemann mapping argument as (z, w, ζ).

Now, it is easy to verify that for $\delta > 0$, if $(z, w, \zeta) \in M$, $(z, w - \delta, \zeta) \in D$. Let $\phi(w, \zeta, \theta, \delta)$ be the point in $\Gamma_{(w-\delta,\zeta)} \cap (M - \delta)$ whose Riemann mapping argument is θ. By the lemma,

$$\lim_{\delta \to 0} \int_{D_0 \times \Gamma} | \hat f(\phi(w, \zeta, \theta, \delta)) - f(R_{(w-\delta,\zeta)}(e^{i\theta}), w - \delta, \zeta) |^2 dV = 0,$$

or what is the same

$$\lim_{\delta \to 0} \int_M | \hat f(z, w - \delta, \zeta) - f(\psi(z, w - \delta, \zeta)) |^2 dV = 0.$$

Now the mapping $(z, w, \zeta) \to \psi(z, w-\delta, \zeta)$ is a differentiable family of transformations on M, tending to the identity as $\delta \to 0$. Thus

$$\lim_{\delta \to 0} \int_M |f(\psi(z, w - \delta, \zeta)) - f(z, w, \zeta)|^2 dV = 0.$$

Thus $\hat{f}(z, w-\delta, \zeta) \to \hat{f}(z, w, \zeta)$ as $\delta \to 0$ in L^2 on M, and since $M-\delta \subset D$, $\hat{f}(z, w-\delta, \zeta)$ is holomorphic on M.

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