HILBERT NULLSTELLENSATZ IN GLOBAL COMPLEX-ANALYTIC CASE

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It is well known that the Hilbert Nullstellensatz holds in the algebraic case [3, Theorem 14, p. 164] as well as in the local complex-analytic case [2, III.A.7]. The question arises whether it holds in the global complex-analytic case when we have a Stein space. In this paper this question is answered in the affirmative for most prime ideals, whereas the answer is negative in the general case.

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Complex spaces here are all in the sense of Grauert [1, Section 1].

Suppose $(X, \mathcal{H})$ is a complex space. By a *holomorphic function* on $(X, \mathcal{H})$ we mean an element of $\Gamma(X, \mathcal{H})$. Suppose f is a holomorphic function on $(X, \mathcal{H})$ and $x \in X$. We say that f *vanishes* at x if the germ f_x is not a unit in the local ring $\mathcal{H}_x$. f *vanishes* on a subset E if f vanishes at every point of E. Suppose M is a set of holomorphic functions on $(X, \mathcal{H})$. *By the analytic subvariety Z of X defined by M* we mean the set

$$Z = \{x \mid x \in X, f \text{ vanishes at } x \text{ for all } f \in M\}.$$

LEMMA. *Suppose M is a set of holomorphic functions on $(X, \mathcal{H})$ and Z is the analytic subvariety of X defined by M. Suppose $x \in X$. If f is a holomorphic function and vanishes on Z, then there exists a natural number k such that $f_x^k = \sum_{i=1}^m \alpha_i (f_i)_x$ for some $f_i \in M$ and $\alpha_i \in \mathcal{H}_x$, $1 \leqq i \leqq m$.*

PROOF. There exists an open neighbourhood U of x in X and $f_1, \cdots, f_m \in M$ such that

(i) U is a complex subspace of an open subset G of C^N and $\mathcal{H} \mid U = ({}_G\mathcal{O} / \sum_{j=1}^n {}_G\mathcal{O} g_j) \mid U$ for some holomorphic functions $g_1, \cdots, g_n$ on G, where ${}_G\mathcal{O}$ is the structure-sheaf of G,

(ii) $U \cap Z = \{y \in U \mid f_i \text{ vanishes at } y \text{ for } 1 \leqq i \leqq m\}$, and

(iii) the restrictions of $f, f_1, \cdots, f_m$ to U are induced by holomorphic functions $\tilde{f}, \tilde{f}_1, \cdots, \tilde{f}_m$ on G. It is readily checked that $U \cap Z = \{z \in G \mid \tilde{f}_i(z) = 0,\ 1 \leqq i \leqq m, \text{ and } g_j(z) = 0,\ 1 \leqq j \leqq n\}$ and $\tilde{f}$ vanishes on $U \cap Z$. By the Hilbert Nullstellensatz in the local complex-

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analytic case [2, III.A.7] there exist $\beta_i \in {}_G\mathcal{O}_x$, $1 \leqq i \leqq m$, and $\gamma_j \in {}_G\mathcal{O}_x$, $1 \leqq j \leqq n$, such that $\bar{f}_x^k = \sum_{i=1}^m \beta_i(\bar{f}_i)_x + \sum_{j=1}^n \gamma_j(g_j)_x$ for some natural number k. Let $\alpha_i \in \mathcal{H}_x$ be induced by β_i, $1 \leqq i \leqq m$. Then $f_x^k = \sum_{i=1}^m \alpha_i(f_i)_x$. q.e.d.

THEOREM (HILBERT NULLSTELLENSATZ FOR PRIME IDEALS). *Suppose $(X, \mathcal{H})$ is a Stein space and R is the ring of holomorphic functions on $(X, \mathcal{H})$. If P is a prime ideal in R and Y is the nonempty analytic subvariety of X defined by P, then P is precisely the set of all holomorphic functions on $(X, \mathcal{H})$ which vanish on Y.*

PROOF. Let Q be the set of holomorphic functions on $(X, \mathcal{H})$ which vanish on Y. It is obvious that $P \subset Q$. To prove $Q \subset P$, suppose $f \in Q$. Take $x \in Y$. Then by the Lemma there is a natural number k such that $f_x^k = \sum_{i=1}^m \alpha_i(f_i)_x$ for some $f_i \in P$ and $\alpha_i \in \mathcal{H}_x$, $1 \leqq i \leqq m$. Let $\mathcal{G} = (\sum_{i=1}^m \mathcal{H}f_i) : f^k$, i.e. $\mathcal{G}$ is the subsheaf of $\mathcal{H}$ defined by

$$\mathcal{G}_y = \left\{ s \mid s \in \mathcal{H}_y,\ sf_y^k \in \left(\sum_{i=1}^m \mathcal{H}f_i \right)_y \right\} \quad \text{for } y \in X.$$

$\mathcal{G}$ is a coherent ideal-sheaf on $(X, \mathcal{H})$, because $\mathcal{G}$ is the kernel of the sheaf-homomorphism $\phi\psi$, where $\psi: \mathcal{H} \to \mathcal{H}$ is defined by $\psi(s) = sf_y^k$ for $s \in \mathcal{H}_y$ and $y \in X$ and $\phi: \mathcal{H} \to \mathcal{H}/\sum_{i=1}^m \mathcal{H}f_i$ is the quotient map. Obviously $\mathcal{G}_x = \mathcal{H}_x$. Suppose 1_x is the identity element of the local ring $\mathcal{H}_x$. By Cartan's Theorem A [1, Section 2, Satz 4], $1_x = \sum_{j=1}^n \beta_j(g_j)_x$ for some $\beta_j \in \mathcal{H}_x$ and $g_j \in \Gamma(X, \mathcal{G})$, $1 \leqq j \leqq n$. For some j, $(g_j)_x$ is a unit in the local ring $\mathcal{H}_x$. Let $g = g_j$. Since g does not vanish on Y, $g \notin P$. Now we have $f^k g \in \Gamma(X, \sum_{i=1}^m \mathcal{H}f_i)$. Consider the short exact sequence

$$(*) \qquad 0 \to \mathcal{K} \xrightarrow{\mu} \mathcal{H}^m \xrightarrow{\lambda} \sum_{i=1}^m \mathcal{H}f_i \to 0,$$

where λ is defined by $\lambda(0, \cdots, 0, 1_y, 0, \cdots, 0) = (f_i)_y$ (1_y is in the ith position) for $y \in X$, $\mathcal{K}$ = kernel of λ, and μ is the inclusion. By Cartan's Theorem B [1, Section 2, Satz 3], $H^1(X, \mathcal{K}) = 0$. From the cohomology sequence of (*) we conclude that $\Gamma(X, \mathcal{H}^m) \to \Gamma(X, \sum_{i=1}^m \mathcal{H}f_i)$ is surjective. There exist $\gamma_i \in R$, $1 \leqq i \leqq m$, such that $f^k g = \sum_{i=1}^m \gamma_i f_i$. $f^k g \in P$. Since $g \notin P$ and P is prime, $f \in P$. $Q \subset P$. q.e.d.

COROLLARY 1. *Y is irreducible.*

PROOF. Suppose $Y = Y_1 \cup Y_2$, Y_i is subvariety of X and $Y_i \neq Y$, $i = 1, 2$. Let P_i be the ideal of all holomorphic functions on $(X, \mathcal{H})$ which vanish on Y_i, $i = 1, 2$. Then $P = P_1 \cap P_2$. By applying Cartan's

Theorem A [1, Section 2, Satz 4], we see that $P \neq P_i$, $i=1, 2$. P is not prime and we get a contradiction. q.e.d.

COROLLARY 2. *There is a one-to-one correspondence between the set of all irreducible subvarieties of a Stein space and the set of all prime ideals of the ring of holomorphic functions defining nonempty subvarieties.*

COUNTEREXAMPLE. We give a counterexample here to show that the Hilbert Nullstellensatz does not hold in the general case. Suppose $X = \mathbf{C}$. For every natural number n let f_n be a holomorphic function on $\mathbf{C}$ whose zeros form precisely the set $\{m \mid m$ is a natural integer $\geqq n$ or $m=0\}$. Let I be the ideal in the ring of holomorphic functions on $\mathbf{C}$ generated by $\{f_n \mid n$ a natural number$\}$. Let $f=z$. The subvariety Z of $\mathbf{C}$ defined by I is $\{0\}$. f vanishes on Z, but no power of f belongs to I.

REMARK. It is clear from the Hilbert Nullstellensatz for prime ideals that the Hilbert Nullstellensatz holds for an ideal in the ring of holomorphic functions on a Stein space defining a nonempty subvariety if and only if the radical of the ideal is an intersection of prime ideals.

ADDED IN PROOF. The theorem is false if Y is empty as the following example shows. Let $X = \mathbf{C}$. Let F be an ultrafilter in X containing $\{m \mid m$ is a natural number $\geqq n\}$ for every natural number n. Let P be the set of entire functions whose zero-sets are members of F. This example grew out of a conversation with Professor T. Nagano and the author wishes to express his thanks.

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