CORRECTION TO "THE EXISTENCE OF PROPER SOLUTIONS OF A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION"

J. D. SCHUUR

I am grateful to Professor L. K. Jackson for pointing out an error in the proof of the lemma and in the proof of Theorem 2 in my paper [1]. Here I shall give a new version of the lemma and a second proof of the Theorem.

The error arises from ignoring equations with solutions which have a finite maximal interval of existence. In the lemma, such a solution could remain within the triangle T for its entire interval of existence; in Theorem 2, there is a solution that remains in the region T for its entire interval of existence, as guaranteed by Ważewski's Theorem, but that interval might be finite. Jackson provided the example

$$x'' = x + (2 + 8(1 - t)^2)(x')^3 + 2\left|1 - t\right|^{1/2}x' \quad (t \geqq 0)$$

and the solution $x(t) = 1 + (1-t)^{1/2}$ which has $[0, 1)$ as its maximal interval of existence. (In the lemma, if $A = 2$ and $c = 4$, then $M = 2$ and $d = 8.5$.)

LEMMA (NEW VERSION). *Given $A > 0$ and $0 \leqq a < c$ there exists a $d(A, c) > 0$ such that if $x(t)$ is a solution of* (1) *with $0 < x(a) < (-Aa/c) + A$, $x'(a) \leqq -d$, then either $x(b) = 0$ for some $b \in (a, c)$, or $x(t)$ has a finite maximal interval of existence $[a, b) \subset [a, c)$ and $x(t) \to x_0$ $(0 \leqq x_0 < (-Ab/c) + A)$ $x'(t) \to -\infty$ as $t \to b$.*

PROOF. Because of the assumptions on (1), a solution $x(t)$ will have a finite maximal interval of existence only if $x(t) \to \pm\infty$ or $x'(t) \to \pm\infty$ at some finite t.

Let $\tau = \{(t, x)\colon 0 \leqq t \leqq c,\ 0 \leqq x \leqq (-At/x) + A\}$ and let H be the hypotenuse of τ. Let $x(t)$ be a solution of (1) with $0 < x(a) < (-Aa/c) + A$. If $x(t)$ leaves τ, it either (a) crosses H, or (b) crosses the t-axis, say at $t = b$. (This covers the case of $x(t) \to \pm\infty$ at some finite t.) If $x(t)$ remains in τ, it has a finite maximal interval of existence, say $[a, b)$, and either (c) $x'(t) \to +\infty$, or (d) $x'(t) \to -\infty$ as $t \to b$.

In a manner similar to that used in the original lemma, we find a $d(A, c) > 0$ such that $x'(a) \leqq -d$ implies that $x'(t) < -A/c$ for $a \leqq t < b$. Such an $x(t)$ can satisfy neither (a) nor (c).

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NEW PROOF OF THEOREM 2. We delete the last paragraph of the old proof and we cannot use Ważewski's Theorem. Continuing from the bottom of page 596:

Let $W=\{(t, x, y): t=0, x=A, -d\leqq y\leqq 0\}$ where d is determined by the lemma with $a=0$, $c>0$ arbitrary. Let $X\subset W$ be such that if $(x(t), y(t))$ is a solution of (4) with $(0, x(0), y(0))\in X$, then there exists a t_0, $0\leqq t_0<\infty$, such that $(x(t), y(t))$ is defined for $0\leqq t\leqq t_0$ and $(t_0, x(t_0), y(t_0))\in Q$; let $Y\subset W$ be defined like X except that $(0, x(0), y(0))\in Y$ implies that $(t_0, x(t_0), y(t_0))\in R$; and let $Z\subset W$ be defined like X except that $(0, x(0), y(0))\in Z$ implies that $(x(t), y(t))$ is defined for $0\leqq t<t_0$ and $x(t)\to x_0\geqq 0$, $y(t)\to -\infty$ as $t\to t_0$.

Solutions of (4) with $(0, x(0), y(0))\in W$ may leave T (the X and Y initial values—included here are solutions with finite maximal intervals of existence and for which $x(t)\to\pm\infty$, or $y(t)\to+\infty$ at some finite t); they may remain in T and have a finite maximal interval of existence (the Z initial values); or they may remain in T and be defined for $0\leqq t<\infty$ (the proper solutions of (1)).

Now X is nonempty since $(0, A, 0)\in X$, $(Y\cup Z)$is nonempty by the lemma, and X and $(Y\cup Z)$ are disjoint. We shall show that X and $(Y\cup Z)$ are open relative to W. It then follows that $W\neq X\cup(Y\cup Z)$ and hence (1) has a proper solution.

Assumption (i) for (1) implies that the solutions of (4) depend continuously upon initial conditions. Let $(0, x_0, y_0)\in X$ and let $(x(t), y(t))$ be the solution of (4) with $x(0)=x_0$, $y(0)=y_0$. There exists a t_1 such that $(x(t), y(t))$ is defined for $0\leqq t\leqq t_1$ and $(t_1, x(t_1), y(t_1))$ is an element of the complement of the closure of T. Let $(0, u_0, v_0)\in W$ and let $(u(t), v(t))$ be the solution of (4) with $u(0)=u_0$, $v(0)=v_0$. We can choose $\delta>0$ such that $|x_0-u_0|+|y_0-v_0|<\delta$ implies that $(u(t), v(t))$ is defined on $[0, t_1]$, $(t_1, u(t_1), v(t_1))$ is an element of the complement of the closure of T, and the point where $(t, u(t), v(t))$ egresses from T lies in Q. Therefore X is open relative to W.

Likewise Y is open relative to W.

Let $(0, x_0, y_0)\in Z$ and let $(x(t), y(t))$ be the solution of (4) with $x(0)=x_0$, $y(0)=y_0$. There exists a t_1, $0<t_1<\infty$, such that $(x(t), y(t))$ exists for $0\leqq t<t_1$ and $x(t)\to x_1$ $(0\leqq x_1<A)$, $y(t)\to-\infty$ as $t\to t_1$.

In the t, x-plane consider the open triangle σ with vertices $(0, 0)$, $(0, 2A)$, $(t_2, 0)$ where t_2 is chosen so that $(t_1, x_1)\in T$. By the lemma, choose $d_1=d_1(2A, t_2)>0$. There exists a t_3, $0<t_3<t_1$, such that $(t_3, x(t_3))\in\sigma$ and $y(t_3)\leqq -2d_1$. Let $(0, u_0, v_0)\in W$ and let $(u(t), v(t))$ be the solution of (4) with $u(0)=u_0$, $v(0)=v_0$. There exists a $\delta>0$ such that $|x_0-u_0|+|y_0-v_0|<\delta$ implies that $(u(t), v(t))$ is defined for $0\leqq t\leqq t_3$, $(t_3, u(t_3))\in\sigma$, and $v(t_3)\leqq -d_1$. Apply the lemma to

$(u(t), v(t))$: there exists a $t_4 \in (t_3, t_2)$ such that either $u(t_4)=0$, or $u(t) \to u_1 \geqq 0$, $v(t) \to -\infty$ as $t \to t_4$. In either case $(0, u_0, v_0) \in (Y \cup Z)$. And since Y is open relative to W it follows that $Y \cup Z$ is open relative to W.

Professor Jackson has also sent me a version of Theorem 2, and he permits $f(t, x, y)$ to be either nondecreasing or nonincreasing in y for each fixed t, x, which he proves using the theory of sub- and superfunctions.

Reference

1. J. D. Schuur, *The existence of proper solutions of a second order ordinary differential equation*, Proc. Amer. Math. Soc. **17** (1966), 595–597.

Michigan State University