THE EXTENDED WEYL INTEGRAL AND RELATED OPERATIONS

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1. **Introduction.** Let the Weyl fractional integral, defined by

$$(1.1)\quad W_\alpha f = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)\,dt, \qquad \alpha > 0,\ 0 < x < \infty,$$

be extended so that $f(t)$ is defined on $(-\infty, \infty)$ and $-\infty < x < \infty$. Then it effects a number of continuous mappings [4, Theorem 1(a); cf. (e), 2(b)], for instance, if $p > 1$, $1/r = 1/p - \alpha > 0$ and $f(t) \in L^p$ $(\equiv L^p(-\infty, \infty))$, a mapping of L^p on L^r; so does the fractional integral

$$(1.2)\quad [W_\alpha^- f](x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1} f(t)\,dt = [W_\alpha\{f(-t)\}](-x).$$

Also $W_\alpha f$ satisfies a number of functional equations, some of which do not hold for the familiar integral $(0 < x < \infty)$.

If $f \in L^1$ and $\alpha < 1$, the forming of the inverse W_α^{-1} of W_α is almost trivial; it is not, however, if $f \in L^p$ $(p > 1)$, owing to the divergence of some integrals. To deal with this problem, the operators

$$(1.3)\quad H_\alpha f = \phi(\alpha)\int_{-\infty}^\infty \frac{|t-x|^\alpha}{t-x} f(t)\,dt,\ K_\alpha f = \phi(\alpha)\int_{-\infty}^\infty |t-x|^{\alpha-1} f(t)\,dt$$

are needed, where $\phi(\alpha) = (2\Gamma(\alpha)\sin\frac{1}{2}\pi\alpha)^{-1}$. They were introduced by Okikiolu [5], [6], $H_\alpha f$ being considered as a modification of the Hilbert transforms $\mathfrak{H}f$. We prove

THEOREM 1. *Let* $p > 1$, $0 < \alpha < 1/p$, $1/r = 1/p - \alpha$, $f \in L^p$, $W_\alpha f = G(x)$, $H_\alpha f = g(x)$, $K_\alpha f = \gamma(x)$, *then* (a) *the inverses* $W_\alpha^{-1}G$, $H_\alpha^{-1}g$, $K_\alpha^{-1}\gamma$ *exist*, (b) *their domains are dense everywhere in* L^r, *but are not closed*, (c) *the operations* W_α^{-1}, H_α^{-1} *and* K_α^{-1} *are not continuous.*

In §3 we will deduce three inversion formulas for W_α, and in §5 the combination of any two of the operators $W_\alpha^\pm$, H_α, K_α, $\mathfrak{H}$ will be discussed, where

$$(1.4)\quad \mathfrak{H}f = \frac{1}{\pi}\,\text{P.V.}\int_{-\infty}^\infty \frac{f(t)}{t-x}\,dt, \qquad \text{P.V.} = \text{principal value}.$$

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Throughout this paper let $1<p<\infty$, $0<\alpha<1/p$, $1/r=1/p-\alpha$. When the two limits of an integral are $-\infty$, ∞ they are omitted.

2. **On Theorem 1.** The proof is based on the formulae

$$(2.1)\quad \begin{aligned} &\text{(i)} \quad \int \frac{|t-\xi|^{\alpha}}{t-\xi}\left(|t|^{-\alpha}-|t-1|^{-\alpha}\right)dt = -2\pi\tan\frac{\pi\alpha}{2}\,\mathfrak{n}(\xi),\\ &\text{(ii)} \quad \int |t-\xi|^{\alpha-1}\left(\frac{|t|^{1-\alpha}}{t}-\frac{|t-1|^{1-\alpha}}{t-1}\right)dt = 2\pi\cot\frac{\pi\alpha}{2}\,\mathfrak{n}(\xi), \end{aligned}$$

where $\mathfrak{n}(\xi)=1$ in $(0, 1)$, $=0$ otherwise, and on the Parseval equations [4], [6]

$$(2.2)\quad \begin{aligned} &\text{(i)} \quad \int k(t)H_\alpha f dt = -\int f(t)H_\alpha k dt,\\ &\text{(ii)} \quad \int k(t)K_\alpha f dt = \int f(t)K_\alpha k dt, \end{aligned}$$

where $f\in L^p$, $k\in L^s$ $(1/s=1-1/r)$, therefore $[H_\alpha f](x)$ and $[K_\alpha f](x)\in L^r$, $[H_\alpha k](x)$ and $[K_\alpha k](x)\in L^{p'}$, as $1/s-\alpha=1-1/r-\alpha$ $=1-(1/p-\alpha)-\alpha=1/p'$. Both equations are deduced from a contour integral of the function

$$(2.3)\qquad F(Z) = (Z-\xi)^{\alpha-1}\{Z^{-\alpha}-(Z-1)^{-\alpha}\} \qquad (Z=t+i\tau)$$

in §4. Using (1.3) and the substitutions $t=\tau/y$, $\xi=x/y$ $(-\infty<y<\infty$, but $y\neq 0)$, we conclude from (2.1) that

$$(2.4)\quad \begin{aligned} &\text{(i)} \quad [H_\alpha(|t|^{-\alpha}-|t-y|^{-\alpha})](x)\\ &\qquad\qquad = -\mathfrak{n}(x/y)\pi\operatorname{sign} y(\Gamma(\alpha)\cos\tfrac{1}{2}\pi\alpha)^{-1},\\ &\text{(ii)} \quad \left[K_\alpha\left(\frac{|t|^{1-\alpha}}{t}-\frac{|t-y|^{1-\alpha}}{t-y}\right)\right](x)\\ &\qquad\qquad = \mathfrak{n}(x/y)\pi\operatorname{sign} y\cot\tfrac{1}{2}\pi\alpha(\Gamma(\alpha)\sin\tfrac{1}{2}\pi\alpha)^{-1}, \end{aligned}$$

taking particular care when $y<0$. Suppose (2.1) to be true. Given that $H_\alpha f=0$ or $K_\alpha f=0$, take $k(t)=|t|^{-\alpha}-|t-y|^{-\alpha}$ or $|t|^{1-\alpha}/t$ $-|t-y|^{1-\alpha}/(t-y)$, respectively, in (2.2). In both cases it follows by (2.4) that, for every real y, $\int_0^y f(t)dt=0$. Hence $f(t)=0$ p.p., therefore both H_α^{-1} and K_α^{-1} exist.

By (2.1) and the linearity of H_α and K_α, every step-function belongs to their ranges which are, therefore, dense everywhere in L^r. If they were closed, each would be L^r, therefore the spaces L^p and L^r would be linearly homeomorphic, though $r>p$, a paradox. The same contradiction would follow if $H_\alpha f$ or $K_\alpha f$ were continuous [1].

To complete the proof with respect to W_α we use the equation (see [**4**, (4.3)])

$$W_\alpha f_1 = H_\alpha(f_1 \sin \tfrac{1}{2}\pi\alpha - \mathfrak{H} f_1 \cos \tfrac{1}{2}\pi\alpha) = H_\alpha f, \tag{2.5}$$

where

$$\begin{aligned} f &= f_1 \sin \tfrac{1}{2}\pi\alpha - \mathfrak{H} f_1 \cos \tfrac{1}{2}\pi\alpha = T f_1; \\ T^{-1} f &= f \sin \tfrac{1}{2}\pi\alpha + \mathfrak{H} f \cos \tfrac{1}{2}\pi\alpha = f_1. \end{aligned} \tag{2.6}$$

Clearly Tf_1 is an endomorphism of L^p. Since $W_\alpha f_1 = H_\alpha f$, W_α has the properties (a), (b), (c) also.

3. **The inversion formulae.** Taking $k(t) = |t|^{-\alpha} - |t-y|^{-\alpha}$ or $|t|^{1-\alpha}/t - |t-y|^{1-\alpha}/(t-y)$ in (2.2) and using (2.4) (i) or (ii), respectively, $H_\alpha f$ or $K_\alpha f = g$, we have

$$\begin{aligned} &\text{(i)} \quad \int_0^y f(t)\,dt = \frac{1}{2\Gamma(1-\alpha)\sin\frac{1}{2}\pi\alpha} \\ &\qquad \cdot \int (|t|^{-\alpha} - |t-y|^{-\alpha}) g(t)\,dt \qquad (H_\alpha f = g) \\ &\text{(ii)} \quad \int_0^y f(t)\,dt = \frac{\sin\frac{1}{2}\pi\alpha}{2\Gamma(1-\alpha)\cos^2\frac{1}{2}\pi\alpha} \\ &\qquad \cdot \int \left(\frac{|t|^{1-\alpha}}{t} - \frac{|t-y|^{1-\alpha}}{t-y} \right) g(t)\,dt \qquad (K_\alpha f = g). \end{aligned} \tag{3.1}$$

These inverses were, in a slightly different form, obtained by Okikiolu. In his proof [**6**] he used two further operators and a number of Fourier transforms. Now we will deal with the inverse of W_α. A formula for K_α is needed which is analogous to (2.5);

$$W_\alpha f = \tan \tfrac{1}{2}\pi\alpha K_\alpha(f \cos \tfrac{1}{2}\pi\alpha + \mathfrak{H} f \sin \tfrac{1}{2}\pi\alpha). \tag{3.2}$$

Clearly the transformation $Uf = f \cos \frac{1}{2}\pi\alpha + \mathfrak{H} f \sin \frac{1}{2}\pi\alpha = f_1$ maps L^p onto itself, with

$$f = U^{-1} f_1 = f_1 \cos \tfrac{1}{2}\pi\alpha - \mathfrak{H} f_1 \sin \tfrac{1}{2}\pi\alpha. \tag{3.3}$$

Since (cf. [**4**, (1.4)])

$$(2 \sin \tfrac{1}{2}\pi\alpha) H_\alpha = W_\alpha - W_\alpha^-, \qquad (2 \sin \tfrac{1}{2}\pi\alpha) K_\alpha = W_\alpha + W_\alpha^-, \tag{3.4}$$

$K_\alpha = -H_\alpha + W_\alpha/\sin \frac{1}{2}\pi\alpha$, and (3.2) follows by (2.5). Thus we gain

THEOREM 1(d). *If $f \in L^p$ and $W_\alpha f = g$, there are the three inversion formulae*

$$\text{(A)}\quad \int_0^x f_1(t)dt = \frac{1}{2\Gamma(1-\alpha)\sin\frac{1}{2}\pi\alpha}\cdot\int(|t|^{-\alpha} - |t-x|^{-\alpha})g(t)dt,$$

$$f = f_1 \sin\tfrac{1}{2}\pi\alpha + \mathfrak{H} f_1 \cos\tfrac{1}{2}\pi\alpha.$$

$$\text{(B)}\quad \int_0^x f_2(t)dt = \frac{\sin\frac{1}{2}\pi\alpha}{2\Gamma(1-\alpha)\cos^2\frac{1}{2}\pi\alpha}\cdot\int\left(\frac{|t|^{1-\alpha}}{t} - \frac{|t-x|^{1-\alpha}}{t-x}\right)g(t)dt,$$

$$f = \cot\tfrac{1}{2}\pi\alpha(f_2\cos\tfrac{1}{2}\pi\alpha - \mathfrak{H} f_2 \sin\tfrac{1}{2}\pi\alpha),$$

$$\text{(C)}\quad \int_0^x f(t)dt = \frac{1}{\Gamma(1-\alpha)}\cdot\left\{\int_0^x t^{-\alpha}g(t)dt + \int_x^\infty (t^{-\alpha} - (t-x)^{-\alpha})g(t)dt\right\} \quad (x > 0),$$

$$= \frac{1}{\Gamma(1-\alpha)}\cdot\left\{-\int_x^0 (t-x)^{-\alpha}g(t)dt + \int_0^\infty (t^{-\alpha} - (t-x)^{-\alpha})g(t)dt\right\} \quad (x < 0).$$

The equations (A) and (B) follow from (3.1) by (2.6) and (3.3), respectively. Since $W_\alpha f = g$, we have $H_\alpha f_1 = K_\alpha \bar{f}_2 = g$ in consequence of (2.5) and (3.4), with

$$f_1 = f\sin\tfrac{1}{2}\alpha\pi - \mathfrak{H} f\cos\tfrac{1}{2}\alpha\pi, \quad \bar{f}_2 = \tan\tfrac{1}{2}\alpha\pi(f\cos\tfrac{1}{2}\alpha\pi + \mathfrak{H} f \sin\tfrac{1}{2}\alpha\pi).$$

To obtain (C), we apply (3.1)(i) to f_1 and (ii) to $\bar{f}_2$ and form the integral of the expression $f_1 \sin^2\frac{1}{2}\alpha\pi + \bar{f}_2\cos\frac{1}{2}\alpha^2\pi$ over $(0, y)$; by (3.5) it is equal to that of $f\sin\frac{1}{2}\alpha\pi$. Thus we arrive at the equation

$$\int_0^y f(t)dt = \frac{1}{2\Gamma(1-\alpha)}\cdot\int\left(|t|^{-\alpha} + \frac{|t|^{1-\alpha}}{t} - |t-y|^{-\alpha} - \frac{|t-y|^{1-\alpha}}{t-y}\right)g(t)dt,$$

which reduces to (C), the result required; clearly the first part of (C) is also the inverse in the familiar case $f(t) \in L^p(0, \infty)$, $x>0$.

4. On the equations (2.1). We start with the case $0<\xi<1$ of (2.1)(i). Let $z=t+i\tau$, $r>0$ and small, $F(z)=(z-\xi)^{\alpha-1}\{z^{-\alpha}-(z-1)^{-\alpha}\}$ (cf. (2.3)), real valued for large real z, and let C_r be the contour consisting of the four semicircles $|z|=1/r$, $=r$, $|z-\xi|=r$, $|z-1|=r$, with $\tau \geqq 0$, and segments of the real axis. Then $F(z)$ is analytic inside and on C_r; on the segments of the real axis which form part of C_r, i.e., on $(-1/r, -r)$, $(r, \xi-r)$, $(\xi+r, 1-r)$, $(1+r, 1/r)$, it is equal, respectively, to the product of $|t-\xi|^{\alpha}/(t-\xi)$ and

$$
\begin{aligned}
&|t|^{-\alpha}-(1-t)^{-\alpha},\ e^{i\pi\alpha}\{t^{-\alpha}-e^{-i\pi\alpha}(1-t)^{-\alpha}\} \\
&\qquad = e^{i\pi\alpha}t^{-\alpha}-(1-t)^{-\alpha},\ t^{-\alpha}-e^{-i\pi\alpha}(1-t)^{-\alpha},\ t^{-\alpha}-(t-1)^{-\alpha}.
\end{aligned}
$$

When $r\to 0$ clearly the integral of $F(z)$ over C_r tends to zero; so does that over the large semicircle, since $0<\alpha<1$ and $|t|^{-\alpha}-|1-t|^{-\alpha} = O(|t|^{-\alpha-1})$ as $|t|\to\infty$. Hence

$$
\begin{aligned}
&\left(\int_{-\infty}^{0}+\int_{1}^{\infty}\right)\frac{|t-\xi|^{\alpha}}{t-\xi}(|t|^{-\alpha}-|t-1|^{-\alpha})dt \\
&\qquad +\int_{0}^{\xi}\frac{|t-\xi|^{\alpha}}{t-\xi}(e^{i\pi\alpha}|t|^{-\alpha}-|t-1|^{-\alpha})dt, \\
&\qquad +\int_{\xi}^{1}\frac{|t-\xi|^{\alpha}}{t-\xi}(|t|^{-\alpha}-e^{-i\pi\alpha}|t-1|^{-\alpha})dt = 0.
\end{aligned}
\tag{4.1}
$$

The difference D between the expression on the left of (2.1)(i) and that in (4.1) is

$$
\begin{aligned}
&-(1-e^{i\pi\alpha})\int_{0}^{\xi}(\xi-t)^{\alpha-1}t^{-\alpha}dt \\
&\qquad + (e^{-i\pi\alpha}-1)\int_{\xi}^{1}(t-\xi)^{\alpha-1}(1-t)^{-\alpha}dt = D.
\end{aligned}
$$

Since the value of each of the two integrals is $\pi/\sin\pi\alpha$, the equation (4.1) reduces to (2.1)(i) provided that $0<\xi<1$.

To deal with the case $\xi>1$ or $\xi<0$ we proceed as before, but arrange the critical points on the real axis in the order 0, 1, ξ or ξ, 0, 1, respectively. The differences corresponding to D turn out to be vanishing identically. Thus (2.1) (i) is true. In a very similar way (2.1) (ii) is proved; the difference $\tilde{D}$ for $0<\xi<1$, analogous to the above D, is

$$(1+e^{i\pi\alpha})\int_0^\xi (\xi-t)^{\alpha-1}t^{-\alpha}dt$$
$$+(1+e^{-i\pi\alpha})\int_\xi^1 (t-\xi)^{\alpha-1}(1-t)^{-\alpha}dt = 2\pi\cot\tfrac{1}{2}\pi\alpha.$$

5. **Combinations of the operators.** Let U_α, U_β be any two of the operators $W^{\pm}$, H, K, $\mathfrak{H}$; let $\alpha>0$, $\beta>0$, $\alpha+\beta<1/p$ and write C for $1/\sin(\alpha+\beta)$, W for $W_{\alpha+\beta}$, W^- for $W^-_{\alpha+\beta}$. We have

THEOREM 2. *Any two of the operations are permutable, and their combination is an operation U or a linear combination of two of them*;

I. $W_\alpha W_\beta f = Wf = W_\beta W_\alpha f;\quad W^-_\alpha W^-_\beta f = W^- f = W^-_\beta W^-_\alpha f;$

II. $W_\alpha W^-_\beta = C\{\sin\pi\alpha W + \sin\pi\beta W^-\} = W^-_\beta W_\alpha;$

III. $H_\alpha W_\beta = C\{\cos\pi(\tfrac{1}{2}\alpha+\beta)W - \cos\tfrac{1}{2}\pi\alpha W^-\} = W_\beta H_\alpha;$

$H_\alpha W^-_\beta = C\{\cos\tfrac{1}{2}\pi\alpha W - \cos\pi(\tfrac{1}{2}\alpha+\beta)W^-\} = W^-_\beta H_\alpha;$

IV. $K_\alpha W_\beta = C\{\cot\tfrac{1}{2}\pi\alpha\sin\pi(\tfrac{1}{2}\alpha+\beta)W + \cos\tfrac{1}{2}\pi\alpha W^-\} = W_\beta K_\alpha;$

$K_\alpha W^-_\beta = C\{\sin\tfrac{1}{2}\pi\alpha W + \cot\tfrac{1}{2}\pi\alpha\sin\pi(\tfrac{1}{2}\alpha+\beta)W^-\} = W^-_\beta K_\alpha;$

V. $$H_\alpha H_\beta = -\frac{1}{2\cos\tfrac{1}{2}\pi(\alpha+\beta)}(W+W^-)$$
$$= -\tan\tfrac{1}{2}\pi(\alpha+\beta)K_{\alpha+\beta} = H_\beta H_\alpha;$$

VI. $K_\alpha K_\beta = \tan\tfrac{1}{2}\pi(\alpha+\beta)\cot\tfrac{1}{2}\pi\alpha\cot\tfrac{1}{2}\pi\beta K_{\alpha+\beta} = K_\beta K_\alpha;$

VII. $H_\alpha K_\beta = \cot\tfrac{1}{2}\pi\beta H_{\alpha+\beta} = K_\beta H_\alpha.$

The results on Hilbert transforms $\mathfrak{H}f$ are obtained by taking the limit $\alpha\to 0$ formally in III, V, VII.

Apparently $H_\alpha f\to\mathfrak{H}f$ as $\alpha\to 0$, cf. (1.3), (1.4); but only formally. The strict proof, however, of the relations gained for $W^{\pm}_\beta\mathfrak{H}$, $H_\beta\mathfrak{H}$, $K_\beta\mathfrak{H}$ from the theorem is not difficult, yet that for $\mathfrak{H}W^{\pm}_\beta$, $\mathfrak{H}H_\beta$, $\mathfrak{H}K_\beta$ is hard and is too long to be inserted here; approximation by entire functions is employed. While I and II are shown by means of Fubini's rule, the other results are reduced to them by (2.5), (2.6), (3.2), (3.4) and their inverses. The details of the proofs are left to the reader.

ADDED IN PROOF (February 20, 1967). Recently Heywood [3] published the following remarkable results. If $0<\alpha<1/p<\infty$, $f\in L^p$ and $H_\alpha f=g$, $H_{-\alpha}g$ exists p.p. as a principal value and $f=-H_{-\alpha}g$; so that $H_{-\alpha}H_\alpha f=-f$ p.p., a striking analogy to the equation $\mathfrak{H}\mathfrak{H}f=-f$. His inversion formula and (3.1)(i) are closely related. For it is easy to show that $\{H_{-\alpha}\mathfrak{e}(t)\}(x)=\{2\Gamma(1-\alpha)\sin\tfrac{1}{2}\pi\alpha\}^{-1}(|x|^{-\alpha}-|1-x|^{-\alpha})$. By the functional equation (2.3)(ii) in [**4**] and by (2.2)(i) we can obtain (3.1)(i) from Heywood's results.

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