OPERATOR REPRESENTATION THEOREMS

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We represent certain types of bounded linear operators from a Banach space into a space BC(S) where S is an arbitrary topological space and BC(S) is the space of bounded continuous scalar-valued functions on S with the sup norm. A comprehensive treatment of work to date may be found in [**1**] and [**7**]. We represent the completely continuous (not to be confused with compact) operator, which has not been considered previously. This yields a sufficient condition for $T: X \to \text{BC}(S)$ to be strictly singular.

DEFINITIONS. (a) A set $U \subset X$ is said to be absolutely convex if it is convex and $\lambda U \subset U$ for $|\lambda| \leqq 1$. (b) We denote the smallest closed absolutely convex set containing W by aco(W). Let $U \subset X$. By U_0 (the polar of U) we shall mean $\{x^* \mid |x^*(u)| \leqq 1 \text{ for all } u \in U\}$. (c) If $U \subset X^*$ we define $U_0 = \{x \mid |u(x)| \leqq 1 \text{ for all } u \in U\}$. We write $U^0{}_0$ for $(U^0)_0$.

The Mackey topology on X^* is that topology generated by the polars of all absolutely convex, weakly compact sets of X [**6**, p. 173].

A bounded linear operator $T: X \to Y$ is said to be completely continuous if it maps weakly convergent sequences into norm convergent sequences. (This is the original definition of Hilbert, which coincides with compactness when X is reflexive.)

LEMMA 1. *Let $A \subset X^*$. If A is Mackey conditionally compact, then given any sequence $\{x_n\} \subset X$ such that $x_n \to x$ in the weak topology, it follows that $x_n \to x$ uniformly on A.*

PROOF. Suppose $\epsilon > 0$ and $x_n \to 0$ weakly. Then $\{x_n, 0\}$ is weakly compact. By the Krein-Smulian theorem [**1**, p. 434], $V = \text{aco}\{x_n, 0\}$ is also weakly compact. By hypothesis there exist x_i^*, $i = 1, \cdots, k$ such that

$$A \subset \bigcup_1^k \left(x_i^* + \frac{\epsilon}{4} V^0 \right).$$

Picking N so large that $|x_i^*(x_n)| < \epsilon/2$ for $n > N$ and $i = 1, \cdots, k$ the result follows.

The lemma above is a corollary in [**4**, p. 134].

The following lemma is due to Elton Lacey.

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LEMMA 2. *The following are equivalent for $T: X \to Y$:*

(a) *T is completely continuous.*

(b) *T^* is continuous as a map from the bounded sets of Y^* with the weak* topology to X^* with its Mackey topology.*

(c) *If $B \subset X^*$ is weak* compact then $T^*(B)$ is Mackey compact.*

PROOF. (a) implies (b). Let V be an absolutely convex weakly compact set in X. It suffices to show that there exists a finite set $\{y_1, \cdots, y_n\} \subset Y$ such that $T^*(W) \subset V^0$ where $W = S(Y^*) \cap a(\{y_1, \cdots, y_n\})^0$ and $S(Y^*)$ is the unit ball in Y^*. Since T is completely continuous, TV is compact. If $\{y_1, \cdots, y_n\}$ is an ϵ-net for TV and $a = 1 - \epsilon$, then the corresponding W is seen to work.

(b) implies (c) is clear.

(c) implies (a) follows from [4, p. 132].

Henceforth X and Y denote Banach spaces and T a bounded linear operator. When we refer to a BC(S) space, it is as mentioned in the introduction.

THEOREM 3. *Let $T: X \to \mathrm{BC}(S)$. $Tx(s) = p(s)x$ where $p: S \to X^*$ is weak* continuous and $p(S)$ is bounded. T is completely continuous if and only if $p(S)$ is Mackey conditionally compact.*

PROOF. The representation of any bounded map by such a $p(\cdot)$ is shown in [7].

Now suppose $p(S)$ is Mackey conditionally compact. We show that T maps weak Cauchy sequences into norm Cauchy sequences. Let $\{x_n\} \subset X$ be weak Cauchy and let $\{n_j\}$ be any increasing sequence of integers. Then $\{x_{n_{j-1}} - x_{n_j}\} \to 0$ weakly. By Lemma 1, given $\epsilon > 0$, there exists $N(\epsilon)$ such that for $n_j > N$

$$\sup_{s \in S} | p(s)(x_{n_{j-1}} - x_{n_j}) | < \epsilon = \sup_{s \in S} | T(x_{n_{j-1}} - x_{n_j})(s) | ,$$

i.e., for every $\{n_j\}$, $\{T(x_{n_{j-1}} - x_{n_j})\} \to 0$ in norm. Thus $\{Tx_n\}$ is Cauchy in norm.

Conversely, suppose T is completely continuous. By Lemma 2, T^* takes weak* compact sets into Mackey compact sets. We have $p = T^* \circ \pi$ where $\pi: S \to \mathrm{BC}(S)^*$ is defined by $\pi(s)f = f(s)$. $\pi(S)$ is contained in the unit ball of $\mathrm{BC}(S)^*$ which is weak* compact; hence, $p(S) = T^* \circ \pi(S)$ is Mackey conditionally compact.

COROLLARY 4. *If S is a compact topological space, then $T: X \to C(S)$ is completely continuous if and only if p is continuous as a map from S with its own topology to X^* with the Mackey topology.*

PROOF. If p is continuous in the Mackey topology, then since S is compact, $p(S)$ is Mackey compact. Thus, T is a fortiori completely continuous.

If T is completely continuous, using Lemma 2 we see that T^* is a weak*-Mackey continuous map on bounded sets. Thus, T^* is weak*-Mackey continuous on $\pi(S)$. But π is a continuous map between S with its own topology and $\pi(S)$ with the weak* topology. Therefore, $p = T^* \circ \pi$ is continuous in the Mackey topology.

DEFINITION. $T: X \to Y$ is said to be strictly singular if whenever T has a bounded inverse on a subspace $M \subset X$, then M is finite dimensional [2, p. 76].

COROLLARY 5. *Let* T: $X \to \mathrm{BC}(S)$; *then, if* $p(S)$ *is weakly conditionally compact and Mackey conditionally compact,* T *is strictly singular.*

PROOF. The weak conditional compactness guarantees [7] that T is weakly compact. In addition, by Theorem 3, T is completely continuous. Any such operator is strictly singular, for suppose T has a bounded inverse on M. M is then reflexive since T is weakly compact. $S(M)$ is then weakly compact. By the complete continuity of T, $TS(M)$ is compact and, since T has a bounded inverse, $S(M)$ is compact. Thus M is finite dimensional.

We do not obtain all of the strictly singular operators in this manner, as the following example shows.

Let $T: l^1 \to c_0$ be given by $T((\alpha_i)) = (\sum_{j=i}^{\infty} \alpha_j)$. Taking the standard basis $\{e_n\}$ in l^1 we see that T is not weakly compact; however, T is strictly singular [3]. Let A be the isomorphism between c_0 and c. Then $A \circ T$: $l^1 \to c$ is not weakly compact but is a strictly singular map into a $C(S)$ space (S is the one-point compactification of the integers). But $A \circ T$ is clearly completely continuous since norm and weak sequential convergence are the same in l^1 [1, p. 296].

If we restrict X, it is possible to weaken the conditions on p and still retain strict singularity. A Banach space X is said to be *infrareflexive* if every infinite dimensional closed subspace M contains an infinite dimensional reflexive subspace. It is known that this class of spaces contains properly the class of quasi-reflexive spaces. For this and related results, see [5].

COROLLARY 6. *Let* T: $X \to C(S)$ *where* X *is infrareflexive and* S *is a compact topological space. If* p *is Mackey continuous, then* T *is strictly singular.*

PROOF. It suffices to show [2, p. 84] that for any infinite dimensional subspace M, there exists $N \subset M$, N infinite dimensional such that T_N is compact. Let M be one such subspace. By hypothesis, M contains N infinite dimensional and reflexive. Thus $S(N)$ is weakly compact. Since p is Mackey continuous given $\epsilon > 0$ and s_α, a net in S converging to s, it follows that there exists α_0 such that for $\alpha > \alpha_0$

$$|(p(s_\alpha) - p(s))x| < \epsilon$$

for $x \in S(N)$. That is, p_N is norm continuous and, by [7, Corollary 8], T_N is compact.

We now apply Lemma 1 to the following result from [1, p. 503].

THEOREM. *Let (S, Σ, μ) be a σ-finite measure space and $T: L_1(S, \Sigma, \mu) \to X^*$, X separable. Then there exists a μ-essentially unique function $x^*(\cdot)$ on S to X^* such that for each x, $x^*(\cdot)x \in L_\infty(S, \Sigma, \mu)$ and*

$$(Tf)x = \int_S x^*(s)xf(s)\mu(ds), \qquad f \in L_1(S, \Sigma, \mu),$$

$$\|T\| = \operatorname*{ess\,sup}_{s \in S} \|x^*(s)\|.$$

If we let $V = \{x^*(\cdot)x \mid \|x\| < 1\}$, then

COROLLARY 7. *With T as above, T is completely continuous if and only if V is Mackey conditionally compact.*

PROOF. Suppose V is Mackey conditionally compact. To show T is completely continuous, it suffices to show that if $f_n \to 0$ weakly, then $Tf_n \to 0$ in norm (see the proof of Theorem 3). Let $\{f_n\}$ be one such sequence. By the hypothesis and Lemma 1, $f_n \to 0$ uniformly on V, i.e., given $\epsilon > 0$ there exists $N(\epsilon)$ such that for $n > N$

$$|(Tf_n)x| = \left|\int_S x^*(s)xf_n(s)\mu(ds)\right| < \epsilon \quad \text{for } x, \quad \|x\| \leqq 1.$$

Therefore, $Tf_n \to 0$ in norm.

Conversely, we note that $V = T^* \cdot JS(X)$ for $(Tf)x = Jx(Tf) = T^*(Jx)f = \int x^*(s)xf(s)\mu(ds) = x^*(\cdot)x(f)$. Since T is completely continuous, by Lemma 2 we have that V is Mackey conditionally compact.

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BIBLIOGRAPHY

1. N. Dunford and J. Schwartz, *Linear operators, Part* I, Interscience, New York, 1958.

2. S. Goldberg, *Unbounded linear operators*, McGraw-Hill, New York, 1966.

3. S. Goldberg and E. O. Thorp, *On some open questions concerning strictly singular operators*, Proc. Amer. Math. Soc. **14** (1963), 334–336.

4. A. Grothendieck, *Sur les applications linéaires faiblement compacts d'espaces du type* $C(K)$, Canad. J. Math. **5** (1953), 129–173.

5. R. Herman and J. Whitley, *An example concerning reflexivity*, Studia Math. **28** (1967), 289–294.

6. J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, N. J., 1963.

7. E. O. Thorp and R. J. Whitley, *Operator representation theorems*, Illinois J. Math. **9** (1965), 595–601.

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