NOTE ON COLLAPSING $K\times I$ WHERE K IS A CONTRACTIBLE POLYHEDRON[1]

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The dunce hat D is obtained from the two-simplex $\langle a, b, c\rangle$ by identifying all three sides, $\langle a, b\rangle = \langle a, c\rangle = \langle b, c\rangle$. D is of interest because it is one of the simplest contractible polyhedra which is not collapsible (there is no free face from which to begin the collapsing). However, it is well known [2] that $D\times I$ is collapsible. This leads to the following conjecture.

CONJECTURE. If K is a contractible two-complex, then $K\times I$ is collapsible. This conjecture is of particular interest since it implies the 3-dimensional Poincaré conjecture [2].

In this note we will consider a method for collapsing $K\times I$ for certain contractible polyhedra K. This method is summarized in the following theorem.

THEOREM. *If L is a collapsible polyhedron and L collapses to K by an elementary collapse, then $K\times I$ is collapsible.*

PROOF. Since L collapses to K by an elementary collapse, we have $L = K\cup B^n$ and $B^n\cap K = B^{n-1}$ with $B^{n-1}\subset \text{bdry}(B^n)$. B^n and B^{n-1} are polyhedral n and $n-1$ balls respectively. Then $K\times I$ collapses to

$$(K\times \{0\})\cup (B^{n-1}\times I) = K'.$$

K' is clearly piecewise linearly homeomorphic to L. Thus $K\times I$ is collapsible.

As a trivial corollary of the previous theorem we get

COROLLARY. *If L is collapsible and L collapses to K, then there is an integer p such that $K\times I^p$ is collapsible.*

Since by [1] a homotopically trivial polyhedron has the same simple homotopy type as a point, we have immediately

COROLLARY. *If K is a homotopically trivial polyhedron, then there is an integer p such that $K\times I^p$ is collapsible.*

EXAMPLE 1. The dunce hat, D. Although it is well known that $D\times I$ is collapsible, an application of the above theorem seems to be conceptually simpler than the usual method.

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In Figure 1 we picture a two simplex, two of whose sides have been identified. The identification of a generator of the cone with its base, as indicated by the numbering of the vertices, yields the dunce hat.

We now expand D to the complex $L=D\cup B^3$ where B^3 is the tetrahedron with vertices v_0, v_1, v_3, v_4, L is indicated in Figure 2.

Now we note that L collapses to D (across $\langle v_1, v_3, v_4\rangle$). Moreover it is easily seen that L is collapsible. First collapse B^3 across $\langle v_0, v_1, v_3\rangle$ and then proceed to collapse the two cell $\langle v_0, v_1, v_4\rangle\cup\langle v_0, v_3, v_4\rangle$ across the one cell $\langle v_0, v_3\rangle$. The remaining collapses are obvious. Thus $D\times I$ is collapsible.

EXAMPLE 2. Bing's house with two rooms, H, H is the two-polyhedron pictured in Figure 3.

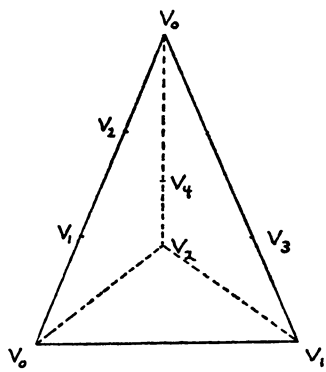

FIGURE 1

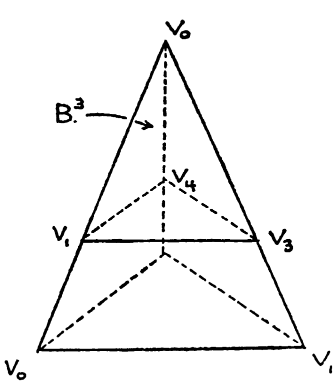

FIGURE 2

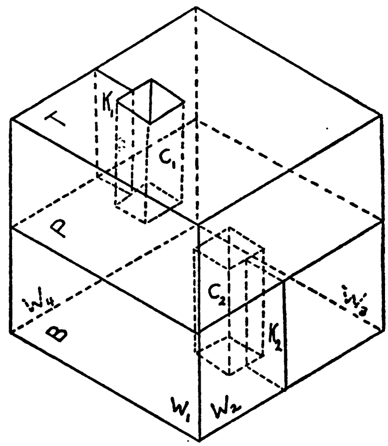

FIGURE 3

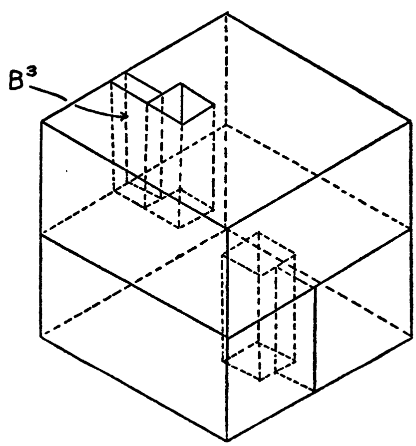

FIGURE 4

In Figure 3 we see that T is a square disk with an open square disk removed, P a square disk with two open square disks removed, and B is a square disk with an open square disk removed. W_1, W_2, W_3 and W_4, the walls of the house, are square disks. C_1 and C_2, the two chimneys, are square cylinders, and K_1 and K_2, the curtains, are rectangular disks.

It is easy to show that H is homotopically trivial, and clearly H is not collapsible.

However, an application of the above theorem shows that $H \times I$ is collapsible. To see this just "fatten" the curtain K_1 up to a 3-cell B^3 as shown in Figure 4.

Let $K = H \cup B^3$. Clearly K collapses to H by an elementary collapse. Moreover, the following steps show that K is collapsible.

1. Collapse B^3 across $B^3 \cap T$.
2. Collapse $B^3 \cap C_1$ across $B^3 \cap C_1 \cap T$.
3. Collapse $B^3 \cap W_4$ across $B^3 \cap W_4 \cap T$.
4. Collapse $B^3 \cap P$ across $B^3 \cap P \cap C_1$.

We have now reached a position where we may collapse across the one cell $B^3 \cap W_4 \cap P$. After several collapses one can eliminate the entire bottom room along with its chimney and curtain. The remaining collapses should be clear.

References

1. J. H. C. Whitehead, *Simplicial spaces, nuclei, and m-groups*, Proc. London Math. Soc. **45** (1939), 243–327.
2. E. C. Zeeman, *On the dunce hat*, Topology **2** (1963), 341–358.

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