L^p BEHAVIOR OF POWER SERIES WITH POSITIVE COEFFICIENTS

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Heywood [4] and others have considered integrability theorems for power series and Laplace transforms that are analogous to known results for Fourier series and transforms. These results are all weighted L^1 results. Here we obtain an L^p theorem which is analogous to the well-known L^p result of Hardy and Littlewood [3].

THEOREM. *Let* $f(x)=\sum_{k=0}^{\infty} a_k x^k$, $a_k \geqq 0$, $0 \leqq x < 1$. *Then for* $1 \leqq p \leqq \infty$,

$$\left[\int_0^1 |f(x)|^p dx\right]^{1/p} < \infty \tag{1}$$

if and only if

$$\left(\sum_{n=1}^{\infty} n^{-2}\left(\sum_{k=0}^{n} a_k\right)^p\right)^{1/p} < \infty. \tag{2}$$

In the analogous theorem for Fourier series the added condition $a_{n+1} \leqq a_n$, $a_n \to 0$, was assumed by Hardy and Littlewood and they stated condition (2) as $\sum n^{p-2} a_n^p < \infty$. Boas [1] has some partial results for Fourier series on L^p theorems with only the condition $a_n \geqq 0$ but only for some L^p spaces with a singular weight function and the question for L^p without a weight seems difficult. For further information see [2].

We first show that if $a_n \geqq 0$, then (1) implies (2). We may assume $1 < p < \infty$, as the two end cases are trivial. Then

$$\begin{aligned}
\int_0^1 [f(x)]^p dx &= \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} [f(x)]^p dx \\
&\geqq \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \left[\sum_{k=0}^{n} a_k x^k\right]^p dx \\
&\geqq \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \left[\sum_{k=0}^{n} a_k \left(1-\frac{1}{n+1}\right)^k\right]^p dx \\
&\geqq A \sum_{n=1}^{\infty} n^{-2}\left[\sum_{k=0}^{n} a_k\right]^p
\end{aligned}$$

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since $(1-1/(n+1))^k \geqq A > 0$, $k=0, 1, \cdots, n$.

In the other direction we have

$$\int_0^1 [f(x)]^p dx = \sum_{n=2}^{\infty} \int_{1-1/(n-1)}^{1-1/n} [f(x)]^p dx$$

$$\leqq A \sum_{n=2}^{\infty} n^{-2} \left[\sum_{k=0}^{\infty} a_k (1-1/n)^k \right]^p$$

$$\leqq A \sum_{n=2}^{\infty} n^{-2} \left[\sum_{k=0}^{\infty} \sum_{j=nk}^{n(k+1)} a_j \left(1 - \frac{1}{n}\right)^j \right]^p$$

$$\leqq A \sum_{n=2}^{\infty} n^{-2} \left[\sum_{k=0}^{\infty} e^{-k} \sum_{j=nk}^{n(k+1)} a_j \right]^p$$

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$$\leqq A \sum_{n=2}^{\infty} n^{-2} \left(\sum_{k=0}^{\infty} e^{-(k/2)p'} \right)^{p/p'} \left(\sum_{k=0}^{\infty} e^{-(kp/2)} \left[\sum_{j=0}^{n(k+1)} a_j \right]^p \right)$$

$$\leqq A \sum_{n=2}^{\infty} n^{-2} \sum_{k=0}^{\infty} e^{-(kp/2)} \left[\sum_{j=0}^{n(k+1)} a_j \right]^p$$

$$= A \sum_{k=0}^{\infty} e^{-(kp/2)} \sum_{n=2}^{\infty} n^{-2} \left[\sum_{j=0}^{n(k+1)} a_j \right]^p$$

$$\leqq A \sum_{k=0}^{\infty} (k+1)^2 e^{-(kp/2)} \sum_{n=2}^{\infty} [(k+1)n]^{-2} \left[\sum_{j=0}^{n(k+1)} a_j \right]^p$$

$$\leqq A \sum_{n=2}^{\infty} n^{-2} \left[\sum_{j=0}^{n} a_j \right]^p.$$

This result can undoubtedly be extended in some of the many ways that Heywood's original L^1 result was extended. We leave these extensions to the interested reader.

Using a result of Konyushkov [5] we have the following corollary.

COROLLARY. *Let $f(x) = \sum a_n x^n$, $a_{n+1} \leqq a_n$. Then for $1 < p < \infty$ we have $f(x) \in L^p(0, 1)$ if and only if $f(e^{i\theta}) \in L^p(-\pi, \pi)$.*

This follows since $f(e^{i\theta}) \in L^p$ if and only if $\sum a_n^p n^{p-2} < \infty$ by the Hardy-Littlewood Theorem and by Konyushkov's result, $\sum a_n^p n^{p-2} < \infty$ if and only if $\sum n^{-2} [\sum_{k=0}^{n} a_k]^p < \infty$. Each of these results also holds for quasi-monotone sequences and thus so does the corollary. That $f(e^{i\theta}) \in L^p(-\pi, \pi)$ implies $f(x) \in L^p(0, 1)$ is a classical result and requires no condition on the coefficients. It does not seem obvious that this is reversible with monotone coefficients.

REFERENCES

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