A MOMENT THEOREM

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Let G be a separable, locally compact, topological abelian group. We denote by $C_c = C_c(G)$ the space of continuous functions with compact support on G and by C_c^+ the set of nonnegative real-valued functions in C_c, excluding the one which is identically zero. $\mathfrak{M}^+$ is taken to be the class of nonnegative, locally finite Borel measures on G, and the topology on $\mathfrak{M}^+$ is that of weak* convergence on compact sets of G. To say that a function $g \geqq 0$ belongs to $\mathfrak{M}^+$ is to say that it is locally integrable with respect to Haar measure dx and therefore the density of $g dx \in \mathfrak{M}^+$. With this convention we will consider G^+, the set of positive continuous characters on G, to be a subset of $\mathfrak{M}^+$ with the induced topology. So far as we know, the following theorem is new, even for G the real line or integers.

THEOREM 1. *Let $f \geqq 0$ be a function on G which has the property that for each $\phi \in C_c^+$ there exists $\mu \in \mathfrak{M}^+$ with $\phi * \mu = f$. Then there exists a nonnegative Borel measure ν on G^+ such that*

$$f(x) = \int_{G^+} g_0(x) \nu(dg_0) \qquad (x \in G). \tag{1}$$

A function satisfying the hypotheses of the theorem will be called *divisible.*

REMARK 1. One can prove that if in addition the group G is *divisible* (for each $x \in G$ and integer $n > 0$ there exists $y \in G$ with $ny = x$), then ν is unique. Otherwise this need not be so. For example, when G = integers, G^+ = real line, then $\nu_1(dx) = \exp(-x^2)dx$ and $\nu_2(dx) = (\exp(-x^2))(1 - \sin(2\pi x))dx$ have the same transforms (1) (see [4, p. 22]).

REMARK 2. The converse to Theorem 1 is also true. Given $\phi \in C_c^+$, define $\hat{\phi}(g_0) = \int_G g_0(-x)\phi(x)dx$, and let $\nu_0(dg_0) = \nu(dg_0)/\hat{\phi}(g_0)$. One checks easily that $g(x) = \int_{G^+} g_0(x)\nu_0(dg_0)$ satisfies $\phi * g = f$.

To establish the representation (1) we will apply a device of Furstenberg together with the Choquet-Bishop-de Leeuw theorem (in its weakest form) to some rather simple observations. Motivation for the theorem itself has come from our study of [1].

LEMMA 1. *For any $f \geqq 0$ on G and $\phi \in C_c^+$, the set*

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$$B_\phi = \{\mu \in \mathfrak{M}^+ \mid \phi*\mu = f\}$$

is compact.

PROOF. Since $\phi \not\equiv 0$, there exists a $\delta > 0$ such that $U = \{x \mid \phi(x) > \delta\}$ is nonempty. Thus it is clear for any $\mu \in B_\phi$ and $x \in G$ that $\mu(x - U) \leqq f(x)/\delta$, and therefore the measures in B_ϕ are locally uniformly bounded. It follows that B_ϕ has compact closure in $\mathfrak{M}^+$. If $\mu = \lim_n \mu_n$, $\mu_n \in B_\phi$, then $\phi*\mu = \lim_n \phi*\mu_n = f$, and $\mu \in B_\phi$. B_ϕ is closed and therefore compact, and the lemma is proved.

LEMMA 2. *Fix $\phi \in C_c^+$. If $\{f_n\}$ is a net of functions on G such that for each n the equation $\phi*\nu = f_n$ has a solution $\nu = \nu_n \in \mathfrak{M}^+$, and if* $\lim_n f_n$ *$= \mu$ exists, then there exists $\nu \in \mathfrak{M}^+$ with $\phi*\nu = \mu$.*

PROOF. Let $\delta > 0$ and U be as in the proof of Lemma 1. There exists a compact subset U_0 of U with nonempty interior and a compact symmetric neighborhood W of e (the identity in G) such that $W + U_0 \subseteq U$. Choose $x \in G$. The assumption $\lim_n f_n = \mu$ guarantees that there is an index n_0 such that if $n > n_0$, then

$$\int_{x+W} f_n(y)dy \leqq \mu(x + W) + 1.$$

Letting $|W|$ be the Haar measure of W there exists $w = w_n \in W$ with

$$f_n(x + w) \leqq (\mu(x + W) + 1)/|W| = \alpha.$$

We conclude as in Lemma 1 that

$$\nu_n(x + w - U) \leqq \alpha/\delta$$

and therefore, since $-U_0 \subseteq w - U$,

$$\nu_n(x - U_0) \leqq \alpha/\delta.$$

The net $\{\nu_n\}$ is locally eventually uniformly bounded, and so it has a convergent subnet $\{\nu_{n_k}\}$. If ν is the limit of this subnet, we have $\phi*\nu = \lim_k \phi*\nu_{n_k} = \mu$, and the lemma is proved.

REMARK 3. Since $\phi*\nu = \mu$, μ has a density f, and our argument shows that $f(x) = \lim_k f_{n_k}(x)$ for each x. Since this is true for any convergent subnet, it must be that $\lim_n f_n(x) = f(x)$, $x \in G$.

LEMMA 3. *Let f be divisible. For each $\phi \in C_c^+$ there exists a divisible function g such that $\phi*g = f$.*

PROOF. If $\Psi = (\psi_1, \cdots, \psi_n)$ is an n-tuple from C_c^+, we denote by $A_\phi = A_\phi(\Psi)$ the subset of B_ϕ consisting of those μ for which the equations

$$\psi_i * \nu_i = \mu \qquad (i = 1, \cdots, n) \tag{2}$$

have solutions $\nu_1, \cdots, \nu_n \in \mathfrak{M}^+$. By Lemma 2, A_ϕ is closed, and we claim $A_\phi \neq \varnothing$. To see this, set $\psi = \psi_1 * \cdots * \psi_n * \phi$, and note that $\psi \in C_c^+$. Since f is divisible, $B_\psi \neq \varnothing$. Select $\nu \in B_\psi$, and define

$$\nu_i = \psi_1 * \psi_2 * \cdots * \hat{\psi}_i * \cdots * \psi_n * \nu, \qquad \mu = \psi_1 * \cdots * \psi_n * \nu$$

where $\hat{}$ denotes omission. Clearly $\mu \in B_\phi$ and $\psi_i * \nu_i = \mu$ for $i = 1, \cdots n$. Thus $\mu \in A_\phi$, and $A_\phi \neq \varnothing$.

From the argument just given we see that the sets $A_\phi(\Psi)$, where Ψ ranges over the finite subsets of C_c^+, have the finite intersection property. By the compactness of B_ϕ there exists a measure $\mu \in B_\phi$ common to all of these sets, and such a measure is obviously given by a density which is a divisible function. The lemma is proved.

We denote by $\Delta = \Delta(G)$ the cone of divisible functions on G. Δ is closed in $\mathfrak{M}^+$ by Lemma 2, and if $\phi \in C_c^+$, we have $\phi * \Delta \subseteq \Delta$. Therefore, by Lemma 3, $\phi * \Delta = \Delta$, $\phi \in C_c^+$.

An *extremal* of Δ is a function g such that if $g = g_1 + g_2$ with $g_1, g_2 \in \Delta$, then both g_1 and g_2 are proportional to g. The following argument is due to Furstenberg ([**1**, especially Theorem 12.2]); it shows that the extremals of Δ are proportional to elements of G^+. Let $g \in \Delta$ be an extremal, and fix $\phi \in C_c^+$. By Lemma 3 there exists $g_1 \in \Delta$ with $\phi * g_1 = g$. This equation expresses g as a "linear combination" of elements of Δ, and therefore the translates of g_1 by negatives of elements in the support of ϕ must be proportional to g. If e is in the support of ϕ, then g_1 is itself proportional to g. Thus, by taking ϕ to have arbitrarily large (compact) support, we find all translates of g proportional to g. It follows that $g = g(e)g_0$, $g_0 \in G^+$. Furstenberg's argument is complete.

To prove Theorem 1 we first determine (using the separability of G) a "cap" of Δ which contains f. This is done by choosing a function $h > 0$, continuous on G, such that $\int_G f(x)h(x)dx \leqq 1$. Then $D = \{g \in \Delta \mid \int_G g(x)h(x)dx \leqq 1)$ is compact, convex and its extreme points are extremals of Δ ([**1**], [**3**]).

Denote by D_e the set of extreme points of D. Since the elements of D_e are either positive multiples of characters or else 0, and since by Remark 3, $\mathfrak{M}^+$ limits in D are pointwise limits, D_e is closed. In this situation an argument based on the Kreĭn-Milman theorem guarantees the existence of a measure $\nu_0 \geqq 0$ on D_e such that

$$f = \int_{D_e} \tilde{g}_0 \nu_0(d\tilde{g}_0). \tag{3}$$

Unless $f \equiv 0$ we may assume ν_0 to be concentrated on $D_e - \{0\}$, and therefore

$$\nu(dg_0) = \tilde{g}_0(e)\nu_0(d\tilde{g}_0) \qquad (g_0 = \tilde{g}_0/\tilde{g}_0(e))$$

defines a measure on G^+. For ν (1) follows from (3), and Theorem 1 is proved.

Uniqueness in (1) is proved for divisible G by substituting $g_0^z(x)$ for $g_0(x)$ in (1) and noting that the entire function arising is the same for any solution ν. Letting z be purely imaginary the mapping $g_0 \to g_0^z$ is a one-one homomorphism from G^+ into $G^\wedge$, the character group of G. In this situation the uniqueness theorem for the Fourier transform implies the uniqueness of ν.

Note. Karlin and Loewner [2] have given a "cone theoretic" characterization of the bilateral Laplace transforms of nonnegative measures on the line. It would be interesting to establish directly the equivalence of their conditions with ours. I thank the referee for pointing this paper out to me.

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