A NOTE ON INTERPOLATION

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I. Let us say that A is a *uniform algebra on* X provided that A is a point separating, uniformly closed subalgebra of $\mathcal{C}(X)$ which contains the constants. The space X is assumed to be a compact Hausdorff space. In this note we establish the following result.

THEOREM. *Let A be a uniform algebra on X and E a closed subset of X. If* $\operatorname{Re} A|E = \mathcal{C}_R(E)$, *then* $A|E = \mathcal{C}(E)$.

This result should be compared with two theorems which appear in the literature. One of these, due to Wermer [**W**], states that *if A is a uniform algebra on X such that* $\operatorname{Re} A$ *is an algebra, then* $A = \mathcal{C}(X)$. The other, due to Hoffman and Wermer [**HW**] (see also [**B**]) says that *if A is a uniform algebra on X, then $A = \mathcal{C}(X)$ if* $\operatorname{Re} A$ *is closed in* $\mathcal{C}_R(X)$. Either of these results applied to $(A|E)^-$ implies that if $\operatorname{Re} A|E = \mathcal{C}_R(E)$, then $A|E$ is dense in $\mathcal{C}(E)$. However, our result does not seem to be a direct consequence of these facts.

The theorem has as a corollary a result which generalizes the Hoffman-Wermer theorem.

COROLLARY. *Let A be a uniform algebra on X. If $E \subset X$ is a closed set such that* $\operatorname{Re} A|E$ *is closed in* $\mathcal{C}_R(E)$, *then* $A|E = \mathcal{C}(E)$.

PROOF OF THE COROLLARY. We have that $(\operatorname{Re} A)|E = \operatorname{Re}[(A|E)^-]$. Thus, the result of Hoffman and Wermer implies that $(A|E)^- = \mathcal{C}(E)$. Consequently $\operatorname{Re} A|E$ is dense in $\mathcal{C}_R(E)$, and since, by hypothesis, it is closed, we have that $\operatorname{Re} A|E = \mathcal{C}_R(E)$. The theorem now implies that $A|E = \mathcal{C}(E)$.

II. **Proof of the Theorem.** By a result of Glicksberg [**G**, Corollary 3.2], we have that $A|E = \mathcal{C}(E)$ if and only if there is a constant C such that $\|\mu_E\| \leqq C\|\mu_{E^c}\|$ whenever μ is a complex regular Borel measure on X which annihilates A. Here μ_E denotes the restriction of μ to E, E^c the complement of E in X.

Suppose no such constant C exists so that for $n = 1, 2, 3, \cdots$ there exists a measure μ^n orthogonal to A which satisfies

$$\|\mu_E^n\| > n\|\mu_{E^c}^n\|. \tag{1}$$

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From this assumption we will derive a contradiction. We may assume that $\|\mu^n\| = 1$.

Let $\mu = \sum_{n=1}^{\infty} 2^{-n}|\mu^n|$. Then $L^\infty = L^\infty(\mu)$ is a commutative Banach algebra with identity under the essential supremum norm and the almost everywhere pointwise operations. Define B to be the algebra of pointwise limits of a.e. $[\mu]$ convergent bounded sequences in A, and let H^∞ denote the norm closure of B in L^∞. This is not the usual definition of H^∞; it was devised so that H^∞ is a closed subalgebra of L^∞ containing A and having the important property that each μ^n annihilates it.

Since the map $f \to \operatorname{Re} f|E$ from A to $\mathcal{C}_R(E)$ is real-linear, continuous, and onto, the open mapping theorem guarantees the existence of a constant C_1 such that if $g \in \mathcal{C}_R(E)$, there is $f \in A$ with $\operatorname{Re} f|E = g$ and $\|f\|_X \leqq C_1\|g\|_E$. From this, it follows that $\operatorname{Re} H^\infty|E = L_R^\infty|E$ and in fact that $\operatorname{Re} B|E = L_R^\infty|E$. Let $g \in L_R^\infty|E$. There exists a bounded sequence $\{g_n\} \subset \mathcal{C}_R(E)$ with $g_n \to g$ a.e. $[\mu]$ on E. Then there is a bounded sequence $\{f_n\} \subset A$ with $\operatorname{Re} f_n|E = g_n$. The sequence $\{f_n\}$ is then a bounded set in $L^2(\mu)$ and so, by passing to a subsequence if necessary, we can assume that f_n converges weakly to $f_0 \in L^2(\mu)$. The Banach-Saks theorem [**RN**, p. 80] now implies that $\|h_k - f_0\|_2 \to 0$ where $h_k = (f_{n_1} + \cdots + f_{n_k})/k$ for some subsequence $\{f_{n_j}\}$ of $\{f_n\}$. A subsequence of $\{h_k\}$ then converges to f_0 pointwise a.e. $[\mu]$. The sequence $\{h_k\}$ is a bounded sequence in A, and thus $f_0 \in B$. Finally, since $\operatorname{Re} f_j|E = g_j$ a.e. $[\mu]$, we see that $h_k|E \to g$ a.e. $[\mu]$, so $\operatorname{Re} f_0|E = g$ a.e. $[\mu]$. Thus, $\operatorname{Re} H^\infty|E = L_R^\infty|E$ as asserted.

Let Γ denote the maximal ideal space of L^∞. Then Γ is an extremally disconnected compact Hausdorff space [**H**, pp. 169–171] and is, therefore, an F-space. (See [**GJ**] for a discussion of F-spaces and extremally disconnected spaces.) The Gelfand transform $\hat{\ }: L^\infty \to \mathcal{C}(\Gamma)$ is an isometric isomorphism with range the whole of $\mathcal{C}(\Gamma)$, and under $\hat{\ }$, H^∞ goes onto a closed subalgebra of $\mathcal{C}(\Gamma)$. The set E corresponds to an open and closed subset $\hat{E}$ of Γ by means of $\hat{\chi}_E = \chi_{\hat{E}}$. Since $\operatorname{Re} H^\infty|E = L_R^\infty|E$, we have that $\operatorname{Re} \hat{H}^\infty|\hat{E} = \mathcal{C}_R(\hat{E})$. Applying either the result of Wermer or that of Hoffman and Wermer cited above, we may conclude that $\hat{H}^\infty|\hat{E}$ is dense in $\mathcal{C}(\hat{E})$. A result of Badé and Curtis [**BC**, Theorem 3.3] now implies that $\hat{H}^\infty|\hat{E} = \mathcal{C}(\hat{E})$, and consequently, we have $H^\infty|E = L^\infty|E$.

Define a function $\phi_n \in L^\infty|E$ by requiring that $|\phi_n| = 1$ a.e. $[\mu]$ on E and that $\phi_n = d\mu^n/d|\mu^n|$ a.e. $[|\mu^n|]$ on E. Since each ϕ_n is of modulus one a.e. $[\mu]$, there is a constant C_2 independent of n such that for each n, there is $f_n \in H^\infty$ with $f_n|E = \phi_n^{-1}$, $\|f_n\| \leqq C_2$. (The existence of the constant C_2 is guaranteed by the open mapping theorem since the map $f \to f|E$ from H^∞ to $L^\infty|E$ is continuous, linear and onto.) Since μ^n is orthogonal to H^∞, we have

$$0 = \int f_n d\mu^n = \int_E \phi_n^{-1} d\mu^n + \int_{E^c} f_n d\mu^n.$$

Thus,

$$\|\mu_E^n\| = \int_E \phi_n^{-1} d\mu^n \leqq \|f_n\|_\infty \|\mu_{E^c}^n\|,$$

whence

$$\|\mu_E^n\| \leqq C_2 \|\mu_{E^c}^n\|. \tag{2}$$

For large values of n, (2) is inconsistent with (1), so we have derived a contradiction and established the theorem.

ADDED IN PROOF. The generalization of Wermer's result **[W]** analogous to our corollary is false without some sort of additional hypothesis. For let X denote the closed unit disc in the complex plane, A the subalgebra of $C(X)$ consisting of functions analytic on the interior of X, E the intersection of X with the real axis. Then $A|E$ is symmetric, hence Re $A|E$ is closed under multiplication, but $A|E \neq C(E)$.

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