ANALYTIC INVARIANTS OF BOUNDED SYMMETRIC DOMAINS[1]

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Introduction. In order to characterize equivalence classes of bounded domains in C^n under holomorphic homeomorphisms S. Bergman introduced various invariants with the help of his kernel function and invariant metric. For general domains one does not have, at least at the moment, a convenient complete system of invariants; for certain very restricted classes of domains, however, it is possible to find such complete systems. K. H. Look [**10**] has shown that within the class of irreducible bounded symmetric domains of classical type each holomorphic equivalence class can be characterized by three constant invariants. In the present paper it will be pointed out that three constants are sufficient also for the class of all irreducible bounded symmetric domains. Beside this slight extension of Look's result we shall also compute certain invariants connected with the Bergman metric in terms of some fundamental invariants for our class. For the special case of the classical domains the invariants of Proposition 3 were also computed in [**10**], using explicit realizations of the domains. The results of [**10**] concerning the "Schwarz constant" were extended to the case of arbitrary symmetric domains in [**9**].

1. Let $M=G/K$ be a noncompact irreducible hermitian symmetric space. It is known [**4**], [**5**] that M has a canonical realization D as a bounded domain in a complex euclidean space V. We denote the complex dimension of M by n, its rank by l.

Denoting by $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$ which is also a Cartan subalgebra of $\mathfrak{k}$, the Lie algebra of K. We denote the set of positive roots of $\mathfrak{g}$ which are not roots of $\mathfrak{k}$ by Φ. It is known [**4**] that Φ has a subset Δ of l strongly orthogonal roots spanning a subalgebra $\mathfrak{h}^-$ of $\mathfrak{h}$, which is a Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. The restrictions of the elements of Φ to $\mathfrak{h}^-$ are of the form α, $(\alpha+\beta)/2$ or $\alpha/2$ ($\alpha, \beta\in\Delta$, $\alpha\neq\beta$), each root of the first type occurring with multiplicity one, each of the second with identical multiplicity a, and each of the third with identical multiplicity b. This follows from the fact [**11**] that the small Weyl group con-

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sists of all signed permutations of Δ. It also follows that the full root system of $(\mathfrak{g}, \mathfrak{k})$ consists of $\pm\alpha$, $(\pm\alpha\pm\beta)/2$, $\pm\alpha/2$ ($\alpha, \beta\in\Delta$, $\alpha\neq\beta$), with the respective multiplicities 1, a, $2b$.

The generalized Cayley transform of D was defined in [**7**]. It induces a splitting of V into a direct sum of euclidean spaces, $V=V_1\oplus V_2$. The dimensions n_1 and n_2 of V_1 and V_2 are then natural numbers canonically associated with D (and M). Clearly, $n_1+n_2=n$. We define the number p by

$$p = (2n_1 + n_2)/l$$

since this combination of constants will frequently occur in what follows.

PROPOSITION 1. *The following relations hold*:
(1) $n_1=al(l-1)/2+l$,
(2) $n_2=bl$,
(3) $p=a(l-1)+b+2$.
The three numbers (l, a, b) *or the three numbers* (l, n_1, n_2) *determine* M *uniquely.*

PROOF. In [**7**] it is shown (Remark after Proposition 4.4) that a basis of V_1 can be constructed from those root vectors $E_{-\gamma}$ ($\gamma\in\Phi$) of $\mathfrak{g}^C$ for which the restriction of γ to $\mathfrak{h}^-$ is of the form α or $(\alpha+\beta)/2$ ($\alpha, \beta\in\Delta$). The total number of these is just $l(l-1)a/2+l$. Similarly the $E_{-\gamma}$ ($\gamma\in\Phi$) with γ restricting to $\alpha/2$ ($\alpha\in\Delta$) span V_2, which proves (2). (3) is immediate from the definition of p.

To see that (l, a, b), or what by (1), (2) is the same, (l, n_1, n_2) determine M uniquely, we have to use the classification of hermitian symmetric spaces.[2] The following list of noncompact irreducible types is by [**5**, Chapter IX] exhaustive and contains no repetitions. The table of constants is easy to compute and our statement can be read off it at once.

I. $SU^r(r+s)/S(U(r)\times U(s))$ ($s\geqq r\geqq 1$), realizable in the space of $r\times s$ complex matrices,

II. $Sp(r, \mathbf{R})/U(r)$ ($r\geqq 2$), realizable in the space of symmetric $r\times r$ matrices,

III.a. $SO^*(4r)/U(2r)$ ($r\geqq 3$), with standard realization as $2r\times 2r$ skew-symmetric matrices,

[2] As Professor S. Helgason has kindly informed me, it is true even in the class of all (not necessarily hermitian) symmetric spaces that the rank and the system of roots with multiplicities determine the space uniquely. However, no *a priori* proof of this seems to be known.

III.b. $SO^*(4r+2)/U(2r+1)$ $(r\geqq 2)$ with a realization as $(2r+1)\times(2r+1)$ skew-symmetric matrices,

IV. $SO^2(n+2)/SO(n)\times SO(2)$ $(n\geqq 5)$, realizable in $\boldsymbol{C}^n$ (cf. [6]),

V. $E_6/SO(10)\times SO(2)$, the 16-dimensional exceptional domain,

VI. $E_7/E_6\times SO(2)$, the 27-dimensional exceptional domain.

Type	n	n_1	n_2	l	a	b	p
I. $(s\geqq r\geqq 1)$	rs	r^2	$r(s-r)$	r	2[3]	$s-r$	$s+r$
II. $(r\geqq 2)$	$r(r+1)/2$	$r(r+1)/2$	0	r	1	0	$r+1$
III.a. $(r\geqq 3)$	$r(2r-1)$	$r(2r-1)$	0	r	4	0	$4r-2$
III.b. $(r\geqq 2)$	$(2r+1)r$	$r(2r-1)$	$2r$	r	4	2	$4r$
IV. $(n\geqq 5)$	n	n	0	2	$n-2$	0	n
V.	16	8	8	2	6	4	12
VI.	27	27	0	3	8	0	18

2. We denote the Bergman kernel of D by $\mathcal{K}(z, w)$. On V we shall use the coordinate system with respect to the orthonormal basis $E_{-\alpha}$ $(\alpha\in\Phi)$. The coefficients of the Bergman metric are

$$g_{\alpha\bar\beta}(z) = \partial^2 \log \mathcal{K}(z, z)/\partial z_\alpha \partial \bar z_\beta. \tag{4}$$

As noted first by S. Bergman, the function

$$I = \mathcal{K}(z, z)/\det g_{\alpha\bar\beta}(z) \tag{5}$$

is clearly invariant under holomorphic homeomorphisms; in the present case, since D is homogeneous, it is constant on D. It follows that for the Ricci tensor we have $r_{\alpha\bar\beta}(z) = -g_{\alpha\bar\beta}(z)$, and that the scalar curvature is $R = -2n$ everywhere on D. The curvature tensor is given by

$$R_{\alpha\bar\beta\gamma\bar\delta}(z) = -\frac{\partial^4 \log \mathcal{K}(z, z)}{\partial z_\alpha \partial \bar z_\beta \partial z_\gamma \partial \bar z_\delta} + g^{\bar\epsilon\tau}(z)\,\frac{\partial^3 \log \mathcal{K}(z, z)}{\partial \bar z_\epsilon \partial z_\alpha \partial z_\gamma}\cdot\frac{\partial^3 \log \mathcal{K}(z, z)}{\partial z_\tau \partial \bar z_\beta \partial \bar z_\delta}, \tag{6}$$

(the summation convention is used). The holomorphic curvature along the vector with coordinates u_α $(\alpha\in\Phi)$ is

[3] If $r=1$, no roots of the second type occur, so $a=0$. In this case also $l=1$, therefore a does not occur explicitly in any of our formulas.

$$k(z, u) = R_{\alpha\bar{\beta}\gamma\bar{\delta}}(z) u_\alpha \bar{u}_\beta u_\gamma \bar{u}_\delta / (g_{\alpha\bar{\beta}}(z) u_\alpha \bar{u}_\beta)^2. \tag{7}$$

We shall use the Koecher Gamma function $\Gamma^*(s)$ associated with D. Its definition is recalled in [**8**]; it can be computed explicitly by a method of S. G. Gindikin [**3**]. After adjusting the volume element used in [**3**, Theorem 2.1] to the normalizations of [**8**] one finds

$$\Gamma^*(s) = (2\pi)^{(n_1 - l)/2} \Gamma(n_1 s/l) \Gamma(n_1 s/l - a/2) \Gamma(n_1 s/l - (l-1)a/2). \tag{8}$$

Proposition 2. *The invariant I has the value*

$$I = \frac{1}{p^n (\operatorname{vol} D)} = \frac{1}{p^n \pi^n} \frac{\Gamma(p)\Gamma(p - a/2) \cdots \Gamma(p - (l-1)a/2)}{\Gamma(1)\Gamma(1 + a/2) \cdots \Gamma(1 + (l-1)a/2)}.$$

Proof. It is sufficient to compute (5) for $z = 0$. By a well-known theorem of E. Cartan, in D there exists a complete orthonormal system of holomorphic functions consisting of homogeneous polynomials. It follows that at the origin $\mathcal{K}(z, w)$ has the expansion

$$\mathcal{K}(z, w) = \mathcal{K}(0, 0)(1 + Q(z, w) + \cdots) \tag{9}$$

where $\mathcal{K}(0, 0)^{-1} = \operatorname{vol} D$, and $Q(z, w)$ is a hermitian quadratic form. $Q(z, w)$ is invariant under the isotropy group K of D, and K is irreducible on V. Hence $Q(z, w) = C \sum_{\alpha \in \Phi} z_\alpha \bar{w}_\alpha$.

To find the value of C we note that by [**8**, Proposition 5.7] we have, for $z' = i \sum_{\alpha \in \Delta} r_\alpha E_{-\alpha}$, $(r_\alpha \geqq 0)$,

$$\mathcal{K}(z', z') = \mathcal{K}(0, 0) \prod_{\alpha \in \Delta} (1 - r_\alpha^2)^{-p} = \mathcal{K}(0, 0)\left(1 + p \sum_{\alpha \in \Delta} r_\alpha^2 + \cdots\right). \tag{10}$$

It follows that $C = p$. Differentiating (9) we have $g_{\alpha\bar{\beta}}(0) = p$ for all $\alpha, \beta \in \Phi$, and hence $\det g_{\alpha\bar{\beta}}(0) = p^n$.

As noted above, $\mathcal{K}(0, 0)^{-1} = \operatorname{vol} D$. By (8) and by [**8**, Proposition 5.7],

$$\operatorname{vol} D = \pi^n \frac{\Gamma(1)\Gamma(1 + a/2) \cdots \Gamma(1 + (l-1)a/2)}{\Gamma(p)\Gamma(p - a/2) \cdots \Gamma(p - (l-1)a/2)}. \tag{11}$$

This finishes the proof of our statement.

Remark 1. For the four classical types of domains vol D was computed by L. K. Hua in [**6**]. His results agree with (11) if one takes into account that in case II his volume element is not the one induced by the quadratic form defining the euclidean structure on V and that in case IV he uses a domain which is our standard D shrunk by a factor $\sqrt{2}$.

Remark 2. (8) and [**8**, Proposition 5.8] also give an explicit expression for the volume of the Bergman-Šilov boundary B of D:

$$\operatorname{vol} B = \frac{2^{(n_1+l)/2}\pi^{(n_1+l)/2+n_2}}{\Gamma(n/l)\,\Gamma(n/l - a/2)\cdots\Gamma(n/l-(l-1)a/2)}.$$

This was also computed in [6] for the classical types; the results there do not in each case agree with the present one. It is clear, however, that e.g. in the case I with $r=1$ (unit ball in $\mathbf{C}^s$) the result in [6] cannot be correct.

PROPOSITION 3. *As z varies over D and u varies over all tangent vectors at z, we have*

$$\inf k(z,u) = -2/p, \quad \sup k(z,u) = -2/pl = -2/(2n_1+n_2).$$

PROOF. Since D is homogeneous it is enough to consider the case $z=0$. Identifying the tangent space at 0 with V, it is also enough to consider the case $u=\sum_{\alpha\in\Delta} u_\alpha E_{-\alpha}$ ($u_\alpha \geqq 0$), since every vector can be transformed to this form by an element of K, which leaves the metric invariant.

As remarked before, we have $\mathcal{K}(z,z)=\sum|\phi_\nu(z)|^2$ where the ϕ_ν are homogeneous polynomials. Hence all derivatives of odd order of $\mathcal{K}(z,z)$ vanish at 0. So, by (6) we have

$$R_{\alpha\bar\beta\gamma\bar\delta}(0) = -\left.\frac{\partial^4 \log \mathcal{K}(z,z)}{\partial z_\alpha \partial \bar z_\beta \partial z_\gamma \partial \bar z_\delta}\right|_{z=0}. \tag{13}$$

For z of the special form $z'=i\sum_{\alpha\in\Delta} r_\alpha E_{-\alpha}$ we have, by (10),

$$\begin{aligned}\log \mathcal{K}(z',z') &= \log \mathcal{K}(0,0) - p\sum_{\alpha\in\Delta}\log(1-r_\alpha^2)\\ &= \log\mathcal{K}(0,0) + p\sum_{\alpha\in\Delta}(r_\alpha^2 + r_\alpha^4/2+\cdots).\end{aligned}$$

Hence, for arbitrary $z\in D$, the sum of those fourth order terms of $\log \mathcal{K}(z,z)$ which involve only the coordinates z_α ($\alpha\in\Delta$) is equal to $(p/2)\sum_{\alpha\in\Delta} z_\alpha^2\bar z_\alpha^2$. For u of the form $\sum_{\alpha\in\Delta} u_\alpha E_{-\alpha}$ we now have, by (13),

$$R_{\alpha\bar\beta\gamma\bar\delta}(0)u_\alpha\bar u_\beta u_\gamma \bar u_\delta = -2p\sum_{\alpha\in\Delta} u_\alpha^4.$$

Earlier we found $g_{\alpha\bar\beta}(0)=p\delta_{\alpha\beta}$ for all α, $\beta\in\Phi$. Hence, by (7),

$$k(0,u) = -\frac{2}{p}\,\frac{\sum_{\alpha\in\Delta} u_\alpha^4}{\left(\sum_{\alpha\in\Delta} u_\alpha^2\right)^2}.$$

As the u_α vary independently over nonnegative numbers, this expression clearly has the lower and upper bounds given in the statement of the proposition.

REMARK. The curvature tensor of the hermitian symmetric manifolds has been studied by different methods by E. Calabi, E. Vesentini and A. Borel [1], [2]. Our computation of sup $k(z, u)$ together with Proposition 2.4(a) and Lemma 2.6 of [1] shows that the eigenvalue λ_1 of the operator Q computed in [1] and [2] is $-2/p$, if the Bergman metric is used (in [1], [2] the metric is normalized differently). The number $\gamma(D)$, which plays a crucial role in [2], is independent of the normalization and turns out to be equal to our p.

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