SPECTRUM OF A SPECTRAL OPERATOR[1]

KIRTI K. OBERAI

1. Preliminaries. Foguel [2] proved that if T is a spectral operator on a Banach space then the spectrum of T coincides with its approximate spectrum. In this paper we show that this is not necessarily so on a locally convex space and we obtain the conditions under which the result is true.

Throughout this paper E will denote a separated, complex locally convex space which we shall assume to be quasi-complete and barreled. Let E' be the dual of E and let $\mathfrak{L}(E)$ be the algebra of all continuous linear maps of E into itself. Let I be the identity map of E onto itself. On $\mathfrak{L}(E)$ we shall always consider the topology τ of uniform convergence on the bounded subsets of E. Also, let Γ be the class of Borel subsets of the complex plane C and let $\hat{C}$ be one point compactification of C.

On a locally convex space the spectrum of an operator $T\in\mathfrak{L}(E)$ is defined in the sense of Waelbroeck [**5**] as follows:

$\lambda\in\hat{C}$ is said to belong to the *resolvent set* $\rho(T)$ of T if and only if there is a neighborhood V_λ of λ in $\hat{C}$ such that there is a function $\mu\to R_\mu$ on $V_\lambda\cap C$ to $\mathfrak{L}(E)$ satisfying, for each $\mu\in V_\lambda\cap C$, the conditions

(i) $R_\mu(\mu I-T)=(\mu I-T)R_\mu=I$;

(ii) $\{R_\mu\colon \mu\in V_\lambda\cap C\}$ is bounded in $\mathfrak{L}(E)$.

The *spectrum*, $\mathrm{sp}(T)$, of T is defined by $\mathrm{sp}(T)=\hat{C}-\rho(T)$. If $\infty\notin\mathrm{sp}(T)$ then T is called a regular element of $\mathfrak{L}(E)$.

The following is the classical definition of spectrum of T.

$\lambda\in C$ is said to belong to the resolvent set $\rho^c(T)$ of $T\in\mathfrak{L}(E)$ if and only if $(\lambda I-T)^{-1}$ exists as an everywhere defined continuous operator on E. The spectrum, $\mathrm{sp}^c(T)$ of T is defined by

$$\mathrm{sp}^c(T) = C - \rho^c(T).$$

The two definitions are equivalent on a Banach space. However, Maeda [**4**] gave examples to show that the two definitions are not necessarily equivalent even on a Fréchet space. We shall call an operator $T\in\mathfrak{L}(E)$ an operator of *type* [B] if the above definitions of the spectrum of T are equivalent. The examples of operators of type

Received by the editors July 18, 1966.

[1] This paper is based on author's dissertation submitted to the University of Illinois for the Ph.D. degree. The author wishes to express his gratitude to Professor R. G. Bartle for his valuable suggestions and guidance. Also, this research was partially supported by the NSF.

[B] are completely bounded operators and compact operators [**4**].

We now give a few definitions. Most of these are based on [**3**].

A function $\{P(\sigma):\sigma\in\Gamma\}$ of commuting projection valued operators defined on Γ with values in $\mathfrak{L}(E)$ is called a *spectral measure* on E if

(a) for each $x\in E$, $P(\cdot)x$ is countably additive in E;

(b) $P(C)=I$;

(c) $\{P(\sigma)\colon \sigma\in\Gamma\}$ is an equicontinuous part of $\mathfrak{L}(E)$.

(Under the assumption that E is barreled condition (c) may be deduced from (a).)

$T\in\mathfrak{L}(E)$ is called a *spectral operator* on E if there exists a (unique) spectral measure $P(\cdot)$ on E such that

(a) $TP(\sigma)=P(\sigma)T$, for all $\sigma\in\Gamma$;

(b) $\mathrm{sp}(T_\sigma)\subset\bar{\sigma}$, $\sigma\in\Gamma$ where T_σ is the restriction of T to $P(\sigma)E$;

(c) for each $x\in E$ and $x'\in E'$, the complex measure $\langle P(\cdot)x, x'\rangle$ has compact support in C.

$S\in\mathfrak{L}(E)$ is called a *scalar operator* on E if there exists a (unique) spectral measure $P(\cdot)$ on E such that for each $x\in E$ and $x'\in E'$, λ is integrable with respect to $\langle P(\cdot)x, x'\rangle$ and

$$\langle Sx, x'\rangle = \int \lambda d\langle P(\lambda)x, x'\rangle.$$

$N\in\mathfrak{L}(E)$ is called *quasi-nilpotent* if for each $x\in E$ and $x'\in E'$, $\lim_{n\to\infty}|\langle N^n, x, x'\rangle|^{1/n}=0$.

If the spectral measure $P(\cdot)$ corresponding to a spectral operator T satisfies the *condition* PC_0 (refer to [**3**]) then T has a unique (canonical) decomposition $T=S+N$ where S is a scalar operator having the same spectral measure as T and N is a quasi-nilpotent operator commuting with S. S is called the *scalar part* of T and N the *radical* of T. In the sequel, whenever we consider a spectral operator T, S, N and $P(\cdot)$, respectively, will denote the scalar part, the radical and the spectral measure corresponding to T. Also T is said to be of *finite type* if $N^n=0$ for some positive integer n.

LEMMA 1. *A net (T_α) in $\mathfrak{L}(E)$ converges to $T\in\mathfrak{L}(E)$ in the topology τ if and only if for each bounded set B in E and each equicontinuous set A in E'*

$$\langle T_\alpha x, x'\rangle \to \langle Tx, x'\rangle, \qquad x\in B,\ x'\in A$$

uniformly on B and A.

PROOF. The topology on E is the topology of uniform convergence on the equicontinuous subsets of E'. Hence $\langle T_\alpha x, x'\rangle\to\langle Tx, x'\rangle$ uniformly on B and A if and only if $T_\alpha x \rightarrow Tx$ uniformly on B, which is true if and only if $T_\alpha \rightarrow T$ in τ.

DEFINITION 1. A spectral measure $P(\cdot)$ on E is said to satisfy the *compactness condition* if for each bounded set B in E and each equicontinuous set A in E' there exists a compact set $\sigma(A, B)$ in C such that

$$\operatorname{Supp}\langle P(\cdot)x, x'\rangle \subset \sigma(A, B) \quad \text{for all } x \in B,\ x' \in A.$$

If T is a spectral operator and $P(\cdot)$ satisfies the compactness condition, then we say that T satisfies the compactness condition. If the spectrum of T is compact then T automatically satisfies the compactness condition. It may be remarked that in this case $P(\cdot)$ also satisfies the condition PC_0.

LEMMA 2. *Let $P(\cdot)$ be a spectral measure on E satisfying the compactness condition. Then $\int \lambda dP(\lambda)$ exists as a Riemann-Stieltjes integral in the topology τ.*

PROOF. Let B, a bounded set in E, and A, an equicontinuous part in E', be fixed. Let σ be a compact set in C such that

$$\operatorname{Supp}\langle P(\cdot)x, x'\rangle \subset \sigma \qquad \text{for all } x \in B \quad \text{and} \quad x' \in A.$$

Let $(\sigma_i)_{i=1}^n$ and $(\delta_j)_{j=1}^m$ be any two subdivisions of σ into disjoint Borel sets of diameters less than any given $\epsilon > 0$. If $\alpha_i \in \sigma_i$ and $\beta_j \in \delta_j$ then

$$\begin{aligned}\sup_{x\in B;\, x'\in A} &\left| \left\langle \sum_i \alpha_i P(\sigma_i)x, x' \right\rangle - \left\langle \sum_j \beta_j P(\delta_j)x, x' \right\rangle \right| \\ &= \sup_{x\in B;\, x'\in A} \left| \left\langle \sum_{i,j} (\alpha_i - \beta_j) P(\sigma_i \cap \delta_j)x, x' \right\rangle \right| \\ &< 2\epsilon \cdot 4K\end{aligned}$$

where $K = \sup |\langle P(\sigma)x, x'\rangle|$, the supremum being taken over all $\sigma \in \Gamma$, $x \in B$ and $x' \in A$. K is finite since $\{P(\sigma) : \sigma \in \Gamma\}$ is an equicontinuous part of $\mathcal{L}(E)$. The lemma now follows from the definition of the integral and Lemma 1.

COROLLARY 1. *If S is a scalar operator which satisfies the compactness condition, then*

$$S = \int \lambda dP(\lambda).$$

The integral exists in the topology τ.

THEOREM 1. *Let T be a spectral operator on E satisfying the compactness condition. Let $P(\{0\})=0$. Then $(TE)^{-}=E$.*

The proof of this theorem may be obtained in a series of lemmas including Lemma 2. Since the proof is very similar to [2, Corollary 2] we omit the details for which the reader may refer to author's dissertation.

LEMMA 3. *Let T be as in Theorem 1. Let σ be a closed set in C and let $\lambda \notin \sigma$. Also, let $(\lambda I-T)x=0$ for some $x \in E$. Then $P(\sigma)x=0$ and $P(\{\lambda\})x=x$.*

PROOF. Since $\lambda \notin \sigma=\bar{\sigma}$, $\lambda \notin \mathrm{sp}(T_\sigma)$. Therefore, $(\lambda I-T_\sigma)^{-1}$ exists as an everywhere defined continuous operator on $P(\sigma)E$ so that

$$(\lambda I - T)_\sigma^{-1}(\lambda I - T)P(\sigma) = P(\sigma).$$

Now

$$P(\sigma)x = (\lambda I - T)_\sigma^{-1}P(\sigma)(\lambda I - T)x = 0.$$

Also

$$P(\{\lambda\})x + \lim_{n\to\infty} P(\{z : |z - \lambda| \geqq 1/n\})x = P(C)x = x.$$

Therefore, $P(\{\lambda\})x=x$.

COROLLARY 2. *If $P(\{0\})=0$ then T is injective.*

PROOF. Suppose $Tx=0$. Let σ be any closed set in C such that $0 \notin \sigma$. By taking $\lambda=0$ in the above lemma we have,

$$0 = P(\{0\})x = x.$$

2. **Classification of spectrum of T.** In this section we shall always assume that T is an operator of type [B] and that E is a locally convex space in which the closed graph theorem holds in the sense that any closed linear operator defined on all of E is continuous. Also, any spectral operator on E will be assumed to satisfy the compactness condition. The spectrum of any $T \in \mathfrak{L}(E)$ is classified as follows:

1. The *point spectrum*, $\mathrm{sp}_p(T)$, of T is the set of all $\lambda \in C$ such that $(\lambda I-T)$ is not injective on E so that $(\lambda I-T)^{-1}$ does not exist.
2. The *continuous spectrum*, $\mathrm{sp}_c(T)$, of T is the set of all $\lambda \in C$ such that $(\lambda I-T)$ is injective on E and the range, $R(\lambda I-T)$, of $(\lambda I-T)$ is a proper dense subset of E.
3. The *residual spectrum*, $\mathrm{sp}_r(T)$, of T is the set of all $\lambda \in C$ such that $(\lambda I-T)$ is injective on E and $R(\lambda I-T)$ is not dense in E.

Since T is an operator of type [B], from the closed graph theorem it follows that $\mathrm{sp}(T)=\mathrm{sp}_p(T)\cup\mathrm{sp}_c(T)\cup\mathrm{sp}_r(T)$, and the sets on the right-hand side are pairwise disjoint.

DEFINITION 2. $\lambda\in C$ is said to belong to the approximate spectrum, $\mathrm{sp}_a(T)$, of T if there exists a continuous seminorm p on E and a net (x_α) in E such that $p(x_\alpha)\geqq 1$ for all α and such that for each continuous seminorm q on E, $q((\lambda I-T)x_\alpha)\to 0$.

It is clear that $\lambda\in\mathrm{sp}_a(T)$ if and only if there is a neighborhood U of 0 in E and a net $(x_\alpha)\subset E\sim U$ such that $(\lambda I-T)x_\alpha\to 0$ in E.

From the above definition, it is not hard to prove the following.

LEMMA 4. (a) $\mathrm{sp}_a(T)\subset\mathrm{sp}(T)$; (b) $\mathrm{sp}_p(T)\subset\mathrm{sp}_a(T)$.

PROPOSITION 1. *Let T be a spectral operator on E and let $\lambda\in\mathrm{sp}(T)$. Then* (a) *if $P(\{\lambda\})=0$ then $\lambda\in\mathrm{sp}_c(T)$* (b) *if $P(\{\lambda\})\neq 0$ and T is of finite type then $\lambda\in\mathrm{sp}_p(T)$.*

PROOF. (a) The operator $(\lambda I-T)$ is a spectral operator on E whose spectral measure $Q(\cdot)$ is defined by $Q(\sigma)=P(\lambda-\sigma)$, so that $Q(\{0\})=P(\{\lambda\})=0$. Hence by Theorem 1, $((\lambda I-T)E)^-=E$. Also, by Corollary 2, $(\lambda I-T)$ is injective. Since $\lambda\notin\rho(T)$, it follows that $\lambda\in\mathrm{sp}_c(T)$.

(b) Let $0\neq x\in P(\{\lambda\})E$ so that $x=P(\{\lambda\})x$. We have,

$$Sx=\int\mu d(P(\mu)x)=\int\mu d(P(\mu)P(\{\lambda\})x)=\int\mu d(P(\mu\cap\{\lambda\})x)$$
$$=\lambda P(\{\lambda\})P(\{\lambda\})x=\lambda x.$$

Since T is of finite type, there is a positive integer n such that $N^nx=0$ and $N^{n-1}x\neq 0$. Therefore, $TN^{n-1}x=(S+N)N^{n-1}x=\lambda N^{n-1}x$. Thus $\lambda\in\mathrm{sp}_p(T)$.

COROLLARY 3. *The residual spectrum of a spectral operator of finite type and, in particular, of a scalar operator is empty.*

PROPOSITION 2. *Let E be complete and let $\lambda\notin\mathrm{sp}_a(T)$. Then $R(\lambda I-T)$ is closed in E.*

PROOF. Let $(y_\alpha)_{\alpha\in D}$ be a net in $R(\lambda I-T)$ and let $y_\alpha\to y$ in E. To prove the proposition we have to show that $y\in R(\lambda I-T)$.

There exist elements z_α in E such that $(\lambda I-T)z_\alpha=y_\alpha$, for all $\alpha\in D$. We assert that (z_α) is a Cauchy net in E. Suppose this is not true so that there exists a neighborhood U of 0 in E, such that for each $\alpha\in D$ there exist $\alpha'\geqq\alpha$ and $\alpha''\geqq\alpha$ in D such that $(z_{\alpha'}-z_{\alpha''})\notin U$. Consider the net $(z_{\alpha'}-z_{\alpha''})_{\alpha\in D}$. We have,

$$(\lambda I-T)(z_{\alpha'}-z_{\alpha''})=y_{\alpha'}-y_{\alpha''}.$$

We shall show that $(y_{\alpha'}-y_{\alpha''})_{\alpha\in D}\to 0$. Let V be any neighborhood of 0 in E. Let W be a circled neighborhood of 0 in E such that $W+W\subset V$. Since $y_\alpha\to y$ there is an $\alpha_V\in D$ such that $(y_\alpha-y)\in W$ for all $\alpha\geqq\alpha_V$. Let $\beta\geqq\alpha_V$ be arbitrary in D. Since $\beta'\geqq\beta$ and $\beta''\geqq\beta$ it follows that $\beta'\geqq\alpha_V$ and $\beta''\geqq\alpha_V$. Hence $(y_{\beta'}-y)\in W$ and $(y_{\beta''}-y)\in W$ so that $y_{\beta'}-y_{\beta''}\in W+W\subset V$, for all $\beta\geqq\alpha_V$. This implies that $(y_{\alpha'}-y_{\alpha''})_{\alpha\in D}\to 0$ in E. Thus, we have proved that $(\lambda I-T)(z_{\alpha'}-z_{\alpha''})\to 0$ while $(z_{\alpha'}-z_{\alpha''})\cap U=\varnothing$. Therefore, by the definition of $\mathrm{sp}_a(T)$, $\lambda\in\mathrm{sp}_a(T)$ which is a contradiction. Hence, (z_α) is a Cauchy net and since E is complete there is a $z\in E$ such that $z_\alpha\to z$; so that

$$y_\alpha = (\lambda I - T)z_\alpha \to (\lambda I - T)z.$$

Since $y_\alpha\to y$, and since E is separated, it follows that $(\lambda I-T)z=y$. Thus, $y\in R(\lambda I-T)$. This proves the proposition.

COROLLARY 4. *Let E be complete. Then* $\mathrm{sp}_c(T)\subset\mathrm{sp}_a(T)$.

PROOF. Let $\lambda\in\mathrm{sp}_c(T)$ so that $\lambda I-T$ is injective and $R(\lambda I-T)$ is dense in E. If $\lambda\notin\mathrm{sp}_a(T)$ then by the above proposition $R(\lambda I-T)$ is closed in E and hence $R(\lambda I-T)=E$. By the closed graph theorem $\lambda\in\rho(T)$, which is a contradiction. Hence, $\lambda\in\mathrm{sp}_a(T)$.

THEOREM 2. *Let E be complete and let T be a spectral operator of finite type on E. Then* $\mathrm{sp}(T)=\mathrm{sp}_a(T)$.

PROOF. By Corollary 3, $\mathrm{sp}_r(T)=\varnothing$. Hence we have,

$$\begin{aligned}\mathrm{sp}_a(T)\subset\mathrm{sp}(T) &= \mathrm{sp}_p(T)\cup\mathrm{sp}_c(T)\\ &\subset \mathrm{sp}_a(T)\cup\mathrm{sp}_a(T)\\ &= \mathrm{sp}_a(T).\end{aligned}$$

This implies that $\mathrm{sp}(T)=\mathrm{sp}_a(T)$.

Foguel [2] proved that if T is a spectral operator on a Banach space then $\mathrm{sp}(T)=\mathrm{sp}_a(T)$. We give a simple example to show that this is not necessarily true even on a Fréchet space.

EXAMPLE. Consider the space E of all sequences $(x_n)_{n=1}^\infty$ where all the x's are complex numbers. We define a countable family p_i, $i=1, 2, \cdots$, of seminorms on E by

$$p_i(x) = \sup_{k\leqq i} |x_k|,$$

where $x=(x_n)$. This family defines a separated, locally convex topology τ on E and (E,τ) is a Fréchet space.

Let I be the identity operator on E and let an operator N on E be defined by

$$Nx = (0, x_1, x_2, \cdots).$$

Then N is a quasi-nilpotent operator on E. Since I is a scalar operator $T=I+N$ is a spectral operator on E. Now $1\in\mathrm{sp}(I)=\mathrm{sp}(T)$. We assert that $1\notin\mathrm{sp}_a(T)$. For, if $1\in\mathrm{sp}_a(T)$ then there is a sequence (x^k) in E such that there is a neighborhood U of 0 in E such that $x^k\notin U$ for all k and such that $(1I-T)x^k=-Nx^k\to 0$. This is not possible. Hence $\mathrm{sp}(T)\neq\mathrm{sp}_a(T)$.

References

1. N. Dunford, *Spectral operators*, Pacific J. Math. **4** (1954), 321–354.

2. S. R. Foguel, *The relation between a spectral operator and its scalar part*, Pacific J. Math. **8** (1958), 51–65.

3. C. Ionescu Tulcea, *Spectral operators on locally convex spaces*, Bull. Amer. Math. Soc. **67** (1961), 125–128.

4. F. Maeda, *Remarks on spectra of operators on a locally convex space*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 1052–1055.

5. L. Waelbroeck, *Locally convex algebras, spectral theory*, Seminar on complex analysis, Institute for Advanced Study, Princeton, N. J., 1958.

University of Illinois and
Queen's University