ON PARTITION FUNCTIONS RELATED TO SCHUR'S SECOND PARTITION THEOREM[1]

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1. **Introduction.** Let $B_d(n)$ denote the number of partitions of n of the form $n=b_1+\cdots+b_s$ with $b_i-b_{i+1}\geqq d$, and if $d|b_i$, then $b_i-b_{i+1}>d$. Let $C_d(n)$ denote the number of those partitions just described subject to the added condition $b_s>d$. These partition functions are associated with certain well-known theorems. The first Rogers-Ramanujan identity [**5**, p. 291] asserts that $B_1(n)$ is equal to the number of partitions of n into parts $\equiv \pm 1 \pmod 5$. The second Rogers-Ramanujan identity asserts that $C_1(n)$ is equal to the number of partitions of n into parts $\equiv \pm 2 \pmod 5$. A theorem proved independently by H. Göllnitz [**3**] and B. Gordon [**4**] asserts that $B_2(n)$ is the number of partitions of n into parts $\equiv 1, 4, 7 \pmod 8$ and that $C_2(n)$ is the number of partitions of n into parts $\equiv 3, 4, 5 \pmod 8$. A theorem of I. J. Schur [**6**] asserts that $B_3(n)$ is the number of partitions of n into parts $\equiv \pm 1 \pmod 6$. For $d>3$, H. L. Alder [**1**, p. 713] has proved that $B_d(n)$ is not equal to the number of partitions of n into parts taken from any set of integers whatsoever.

The object of this paper is to give a proof of Schur's theorem utilizing Watson's q-analog of Whipple's theorem and to give the following result on $C_3(n)$.

THEOREM 3.

$$1+\sum_{n=1}^{\infty} C_3(n)q^n = \prod_{m=1}^{\infty} \frac{(1+q^m)}{(1-q^{6m})} \sum_{n=0}^{\infty} \frac{(-1)^n q^{9n(n+1)/2}(1-q^{6n+3})}{(1+q^{3n+1})(1+q^{3n+2})}.$$

Thus the generating function for $C_3(n)$ is similar to the mock theta functions. Indeed, it is conceivable that a very accurate asymptotic formula for $C_3(n)$ may be found utilizing the techniques developed in [2]. Unfortunately, I have not been able to obtain any simple partition-theoretic interpretation of Theorem 3. To my knowledge nothing at all is known about $C_d(n)$ for $d>3$. I would conjecture, however, that Alder's result for $B_d(n)$ is also valid for $C_d(n)$ with $d>2$.

2. **Proof of theorems.** Let $b_j(m, n)$ denote the number of partitions of n into m parts such that $n=b_1+\cdots+b_m$, $b_i-b_{i+1}\geqq 3$, and if $3|b_i$, then $b_i-b_{i+1}>3$, $b_m>j$.

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LEMMA.

$$b_0(m, n) - b_1(m, n) = b_0(m - 1, n - 3m + 2), \tag{2.1}$$

$$b_1(m, n) - b_2(m, n) = b_1(m - 1, n - 3m + 1), \tag{2.2}$$

$$b_2(m, n) - b_3(m, n) = b_3(m - 1, n - 3m), \tag{2.3}$$

$$b_3(m, n) = b_0(m, n - 3m). \tag{2.4}$$

PROOF. We prove (2.3); the other identities are proved in exactly the same manner. First, $b_2(m, n) - b_3(m, n)$ enumerates the number of partitions of the type enumerated by $b_2(m, n)$ with the added restriction that 3 appears as a summand. Now subtract 3 from every summand. The number of summands is reduced to $m-1$; the number being partitioned is reduced to $n-3m$, and the smallest part now appearing is $\geqq 4$. Hence we now have a partition of the type enumerated by $b_3(m-1, n-3m)$. This procedure establishes a one-to-one correspondence between the partitions enumerated by $b_2(n, m) - b_3(m, n)$ and those enumerated by $b_3(m-1, n-3m)$. Hence (2.3) follows.

Now for $|q| < 1$, we define

$$f_i(x) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_i(m, n) x^m q^n.$$

THEOREM 1.

$$f_0(x) = \prod_{n=0}^{\infty} \frac{(1 + xq^n)}{(1 - x^2 q^{6n})} \sum_{m=0}^{\infty} (-1)^m x^{2m} q^{(1/2)(9m^2 - 3m)} (1 - xq^{6m}) \cdot \prod_{j=0}^{3m-1} \frac{(1 + q^{j+1})}{(1 + xq^j)} \prod_{k=0}^{m-1} \frac{(1 - x^2 q^{6k})}{(1 - q^{6k+6})} .$$

PROOF. (2.1), (2.2), (2.3), and (2.4) imply respectively

$$f_0(x) - f_1(x) = xq f_0(xq^3), \tag{2.5}$$

$$f_1(x) - f_2(x) = xq^2 f_1(xq^3), \tag{2.6}$$

$$f_2(x) - f_3(x) = xq^3 f_3(xq^3), \tag{2.7}$$

$$f_3(x) = f_0(xq^3). \tag{2.8}$$

Thus by (2.5),

$$f_1(x) = f_0(x) - xq f_0(xq^3). \tag{2.9}$$

By (2.7) and (2.8),

$$f_2(x) = f_0(xq^3) + xq^3 f_0(xq^6). \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.6), we obtain

$$f_0(x) = (1 + xq + xq^2)f_0(xq^3) + xq^3(1 - xq^3)f_0(xq^6). \tag{2.11}$$

Let $G(x) = f_0(x)\prod_{n=0}^{\infty}(1 - xq^{3n})^{-1}$. Then by (2.11),

$$(1 - x)G(x) = (1 + xq + xq^2)G(xq^3) + xq^3G(xq^6). \tag{2.12}$$

Thus if $G(x) = \sum_{n=0}^{\infty} A_n(q)x^n$, then $A_0(q) = 1$ and

$$\begin{aligned} A_n(q) - A_{n-1}(q) &= q^{3n}A_n(q) + q^{3n-2}A_{n-1}(q) \\ &\quad + q^{3n-1}A_{n-1}(q) + q^{6n-3}A_{n-1}(q), \end{aligned} \tag{2.13}$$

or

$$\begin{aligned} A_n(q) &= (1 + q^{3n-2} + q^{3n-1} + q^{6n-3})(1 - q^{3n})^{-1}A_{n-1}(q) \\ &= (1 + q^{3n-1})(1 + q^{3n-2})(1 - q^{3n})^{-1}A_{n-1}(q). \end{aligned} \tag{2.14}$$

Thus $A_n(q) = \prod_{j=1}^{n}(1 + q^{3j-1})(1 + q^{3j-2})(1 - q^{3j})^{-1}$. Hence

$$\begin{aligned} f_0(x) &= \prod_{n=0}^{\infty}(1 - xq^{3n}) \sum_{m=0}^{\infty} x^m \\ &\quad \cdot \prod_{j=1}^{m}(1 + q^{3j-1})(1 + q^{3j-2})(1 - q^{3j})^{-1}. \end{aligned} \tag{2.15}$$

In Watson's q-analog of Whipple's theorem [7, p. 100, (3.4.1.5)], first replace q by q^3, then set $a = x$, $e = -q$, $f = -q^2$, and let c, d, and $g \to \infty$. This yields

$$\begin{aligned} &\sum_{m=0}^{\infty} (-1)^m x^{2m} q^{(1/2)(9m^2-3m)} \frac{(1 - xq^{6m})}{(1 - x)} \\ &\qquad \cdot \prod_{j=0}^{3m-1} \frac{(1 + q^{j+1})}{(1 + xq^j)} \prod_{k=0}^{m-1} \frac{(1 - x^2q^{6k})}{(1 - q^{6k+6})} \\ &\quad = \prod_{n=0}^{\infty} \frac{(1 - xq^{3n+3})(1 - xq^{3n})}{(1 + xq^{3n+1})(1 + xq^{3n+2})} \sum_{m=0}^{\infty} x^m \\ &\qquad \cdot \prod_{j=1}^{m}(1 + q^{3j-1})(1 + q^{3j-2})(1 - q^{3j})^{-1} \\ &\quad = f_0(x) \prod_{n=0}^{\infty} \frac{(1 - xq^{3n+3})}{(1 + xq^{3n+1})(1 + xq^{3n+2})}. \end{aligned} \tag{2.16}$$

Simplifying, we obtain Theorem 1.

THEOREM 2.

$$1 + \sum_{n=1}^{\infty} B_3(n) = f_0(1) = \prod_{n=0}^{\infty}(1 - q^{6n+1})^{-1}(1 - q^{6n+5})^{-1}.$$

PROOF. The first equation follows from the definitions. Setting $x=1$ in Theorem 1 and utilizing Jacobi's theorem [**5**, p. 282], we deduce

$$\begin{aligned} f_0(1) &= \prod_{n=0}^{\infty} (1+q^{3n+1})(1+q^{3n+2})(1-q^{3n+3})^{-1} \\ &\quad \cdot \left\{1+\sum_{m=1}^{\infty} (-1)^m q^{(1/2)(9m^2-3m)}(1+q^{3m})\right\} \\ &= \prod_{n=0}^{\infty} (1+q^{3n+1})(1+q^{3n+2})(1-q^{3n+3})^{-1}(1-q^{3n+3}) \\ &= \prod_{n=0}^{\infty} (1+q^{3n+1})(1+q^{3n+2}) \\ &= \prod_{n=0}^{\infty} (1-q^{6n+1})^{-1}(1-q^{6n+5})^{-1}. \end{aligned}$$

One may now prove Schur's partition theorem directly from Theorem 2.

Finally we prove Theorem 3. Utilizing basic definitions, (2.8), and Theorem 1, we have

$$\begin{aligned} 1+\sum_{n=1}^{\infty} C_3(n)q^n &= f_3(1) \\ &= f_0(q^3) \\ &= \prod_{n=1}^{\infty} \frac{(1+q^n)}{(1-q^{6n})} \sum_{m=0}^{\infty} \frac{(-1)^m q^{9m(m+1)/2}(1-q^{6m+3})}{(1+q^{3m+1})(1+q^{3m+2})}. \end{aligned}$$

Thus we have Theorem 3.

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