GENERALIZATIONS OF A PROPERTY OF TCHEBYCHEFF POLYNOMIALS

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The purpose of this paper is to generalize a well-known property of polynomials to functions which are solutions of linear differential equations of the form $L_n y = \sum_{i=0}^{n} p_i(x) y^{(i)} = 0$, where $p_n(x) \neq 0$ and all the coefficients are continuous. In particular, the problem to be generalized is that of finding a polynomial of degree n with leading coefficient 1 whose maximum absolute value deviates least from zero in the interval $-1 \leqq x \leqq 1$ (see for example Courant and Hilbert [1, pp. 88–89]).

First we need:

DEFINITION. The first conjugate point $\eta_1(a)$ of the point a is the smallest number $b > a$ such that there exists a nontrivial solution of $L_n y = 0$ which vanishes at a and has n zeros, counting multiplications, in $[a, b]$.

We can now state the main result of this paper.

THEOREM. *Let* $L_{n+k} y = \sum_{i=0}^{n+k} p_{n+k,i}(x) y^{(i)} = 0$ $(k = 0, 1)$ *be a pair of linear differential equations, where* $p_{n+k,n+k}(x) \neq 0$ $(k = 0, 1)$, $n \geqq 1$ *and all the coefficients are continuous on* $[a, b]$ *and,* $b \leqq \eta_1(a)$ *for* $k = 0, 1$ *(if* $\eta_1(a)$ *does not exist,* b *may be chosen arbitrarily). Suppose that if* $z(x)$ *is a solution of* $L_n y = 0$*, then it is also a solution of* $L_{n+1} y = 0$*. Let* $\{\phi_i(x)\}$ $(i = 1, \cdots, n+1)$ *be a fundamental set of solutions of* $L_{n+1} y = 0$*, where* $\{\phi_i(x)\}$ $(i = 1, \cdots, n)$ *is a fundamental set of solutions of* $L_n y = 0$*. Then the solution* $\phi(x)$ *of* $L_{n+1} y = 0$*, such that* $\phi(x) = \phi_{n+1}(x) + \sum_{i=1}^{n} a_i \phi_i(x)$ *and such that* $\max_{x \in [a,b]} |\phi(x)|$ *is minimal, is that solution* $\phi(x)$ *satisfying*

(i) $\phi(x)$ *has* n *distinct zeros in* (a, b).

(ii) *Let the zeros in* (i) *be located at* $x_1 < x_2 < \cdots < x_n$*. Let the points at which* $\phi(x)$ *attains its extreme values in* $[a, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$, $[x_n, b]$ *be denoted respectively by* $c_1, c_2, \cdots, c_n, c_{n+1}$*. Then* $|\phi(c_1)| = |\phi(c_2)| = \cdots = |\phi(c_{n+1})|$.

PROOF. First it must be shown that such a solution $\phi(x)$ of $L_{n+1} y = 0$ exists. We can always assign n zeros in (a, b) in an arbitrary manner [2, Theorem 3]. We need first to show that these points may be adjusted such that the relative values $|\phi(c_i)|$ $(i = 1, \cdots, n+1)$ may be adjusted at will. Denote by $\|y\|$ the norm $\sum_{i=0}^{n} \max_{x \in [a,b]} |y^{(i)}(x)|$.

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Let $S=\{y|y$ is a solution of $L_{n+1}y=0$ with n distinct zeros on (a, b) and $\|y\|=1\}$. First fix $x_2, x_3, \cdots, x_n$ for $y\in S$ and let x_1 approach a. Then $y(c_1)$ approaches zero since $\|y\|=1$. Similarly $y(c_i)\to 0$ as $x_i-x_{i-1}\to 0$ $(i=2, 3 \cdots, n)$, but $y(c_j)\not\to 0$ $(i\neq j)$ since $\eta_1(a)\geqq b$ (for $L_{n+1}y=0$) implies that no solution has $n+1$ zeros on $[a, b)$ (or $(a, b]$), by [2]. Thus by appropriate choice of $x_1, \cdots, x_n$ we can adjust the relative values of $|y(c_i)|$.

We shall now establish the existence of the solution $\phi(x)$ of the theorem. By choosing x_1 sufficiently near a we can choose a solution $y(x)\in S$ such that $|y(c_1)|\leqq|y(c_i)|$, $i>1$. Now let $x_2\to x_1$ until $|y(c_1)|=|y(c_2)|$, then by choosing both x_1 and x_2 sufficiently close to a we find $|y(c_1)|=|y(c_2)|\leqq|y(c_i)|$, $i>2$. Continuing in this manner we find a solution $y(x)=\Psi_1(x)\in S$ such that $|\Psi_1(c_1)|=|\Psi_1(c_2)| = \cdots =|\Psi_1(c_n)|\leqq|\Psi(c_{n+1})|$. We now shall vary the points $x_1, \cdots, x_n$ as determined by $\Psi_1(x)=y(x)$. Let $x_n\to b$ until $|y(c_{n+1})| =|y(c_j)|$ for some $j<n+1$. Let $c_{j_1}<c_{j_2}<\cdots<c_{j_s}$ be the values such that $|y(c_{n+1})|=|y(c_{j_1})|=|y(c_{j_2})|=\cdots=|y(c_{j_s})|$. If $s=n$ we are of course done, so we shall suppose $s<n$. Let k be the smallest integer for which the sequences $j_1, j_2, \cdots, j_k$ and $1, 2, \cdots, k$ do not coincide. For $j_i<k$ let $x_{j_i}\to x_{j_i-1}$ (and if $j_1=1$ let $x_{j_1}\to a$), for $j_i>k$ let $x_{j_i-1}\to x_{j_i}$ and $x_n\to b$ (that is the zero points to the left of x_k are moved to the left and the zero points to the right of x_k are moved to the right) until $|y(c_{n+1})|=|y(c_{j_1})|=\cdots=|y(c_{j_s})|=|y(c_m)|$ for some $m \notin\{j_1, j_2, \cdots, j_s, n+1\}$. Continuing in this manner we can find a solution $y(x)=\Psi_2(x)\in S$ such that $|\Psi_2(c_{n+1})|=|\Psi_2(c_i)|\geqq|\Psi_2(c_j)|$ for some $j<n+1$ and all $i\neq j$, $i<n+1$. In a manner similar to that used to find $\Psi_1(x)$ we can find a solution $\Psi_3(x)\in S$ such that $|\Psi_3(c_{n+1})| =|\Psi_3(c_i)|\leqq|\Psi_3(c_j)|$ for all $i\neq j$, $i<n+1$. Comparing $\Psi_3(x)$ and $\Psi_2(x)$ we conclude, from the continuous dependence of solutions on $x_1, x_2, \cdots, x_n$, that the solution $\phi(x)$ of the theorem exists. Further, since $\eta_1(a)\geqq b$ (for $L_ny=0$) implies there is no solution of $L_ny=0$ with n zeros in (a, b), we conclude $\phi(x)$ is a solution of $L_{n+1}y=0$ but not of $L_ny=0$. Hence, after multiplying $\phi(x)$ by a nonzero constant if necessary, we may write $\phi(x)=\phi_{n+1}(x)+\sum_{i=1}^n a_i\phi_i(x)$ (we drop the requirement here that the solution be a member of S).

We shall now show that this solution $\phi(x)$ has the desired extremal property. Suppose it does not, then there is a solution $z(x)=\phi_{n+1}(x) +\sum_{i=1}^n b_i\phi_i(x)$ such that $\max_{x\in[a,b]}|z(x)|<\max_{x\in[a,b]}|\phi(x)| =|\phi(c_i)|$ $(i=1, 2, \cdots, n+1)$. We shall for definiteness assume $\phi(a)>0$. Then $\phi(c_1)-z(c_1)>0$, $\phi(c_2)-z(c_2)<0, \cdots$. Hence $\phi(x) -z(x)=\sum_{i=1}^n (a_i-b_i)\phi_i(x)$ vanishes at least once in (c_1, c_2), (c_2, c_3),

$\cdots$, (c_n, c_{n+1}) and hence has n zeros in (a, b). However, $\phi(x)-z(x)$ is then a solution of $L_ny=0$ with n zeros in (a, b) which is impossible since $\eta_1(a)\geqq b$, by [2].

COROLLARY. *Let* $L_{n+k}y=\sum_{i=1}^{n+k} p_{n+k,i}(x)y^{(i)}=0$ $(k=0, 2)$ *be a pair of linear differential equations, where* $p_{n+k,n+k}(x)\neq 0$ $(k=0, 2)$, $n\geqq 1$ *and all the coefficients are continuous on* $[a, b]$ *and* $b\leqq\eta_1(a)$ *for* $k=0,2$ *(if* $\eta_1(a)$ *does not exist, b may be chosen arbitrarily). Suppose that if* $z(x)$ *is a solution of* $L_ny=0$, *then it is also a solution of* $L_{n+2}y=0$. *Let* $\{\phi_i(x)\}$ $(i=1, \cdots, n)$ *be a fundamental set of solutions of* $L_ny=0$. *Let* $\phi_{n+2}(x)$ *be a solution of* $L_{n+2}y=0$ *which has* $n+1$ *distinct zeros in* (a, b) *located at* $x_1<x_2<\cdots<x_{n+1}$ *such that* $\max_{x\in[a,x_1]}|\phi_{n+2}(x)| =\max_{x\in[x_1,x_2]}|\phi_{n+2}(x)|=\cdots=\max_{x\in[x_{n+1},b]}|\phi_{n+2}(x)|$ *(i.e.* $\phi_{n+2}(x)$ *is the solution* $\phi(x)$ *of the theorem). Let the points at which their extreme values are attained be denoted by* $c_1, c_2, \cdots c_{n+2}$ *respectively. Let* $\phi_{n+1}(x)$ *be a solution of* $L_{n+2}y=0$ *which satisfies the conditions* $\phi(c_i)=0$ *for* $i=1, 2, \cdots, n+1$. *Then the solution* $\phi(x)$ *of* $L_{n+2}y=0$, *such that* $\phi(x)=\phi_{n+2}(x)+\sum_{i=2}^{n+1} a_i\phi_i(x)$ *and such that* $\max_{x\in[a,b]}|\phi(x)|$ *is minimal, is* $\phi(x)=\phi_{n+2}(x)$.

PROOF. We first note that $\phi_{n+1}(x)$ is not a solution of $L_ny=0$ since it has $n+1$ zeros on $[a, b)$, hence it is independent of $\phi_i(x)$ $(i=1, 2, \cdots, n)$. Further it is independent of $\phi_{n+2}(x)$ since the zeros are different. Suppose now ϕ_{n+2} does not have the desired extremal property. Then there is a solution $z(x)=\phi_{n+2}(x)+\sum_{i=1}^{n+1} b_i\phi_i(x)$ such that $\max_{x\in[a,b]}|z(x)|<\max_{x\in[a,b]}|\phi_{n+2}(x)|$. We shall assume $\phi_{n+2}(a)>0$. In particular $\phi_{n+2}(c_1)-z(c_1)>0$, $\phi_{n+2}(c_2)-z(c_2)<0, \cdots$, but, since $\phi_{n+1}(c_i)=0$ $(i=1, \cdots, n+1)$, $\phi_{n+2}(c_i)-[z(c_i)-b_{n+1}\phi_{n+1}(c_i)]$ has these same properties for $i=1, \cdots, n+1$. Hence $\phi_{n+2}(x) -[z(x)-b_{n+1}\phi_{n+1}(x)]=-\sum_{i=1}^{n} b_i\phi_i(x)$ vanishes at least once in (c_1, c_2), (c_2, c_3), $\cdots$, (c_n, c_{n+1}) and hence has n zeros in (a, b). But, since $\eta_1(a)\geqq b$ for $L_ny=0$ and $-\sum_{i=0}^{n} b_i\phi_i(x)$ is a solution of $L_ny =0$, we conclude that $b_i=0$ $(i=1, \cdots, n)$. Hence $z(x) =\phi_{n+2}(x)+b_{n+1}\phi_{n+1}(x)$. However $z(c_1)=\phi_{n+2}(c_1)$, contradicting the assumption that $\phi_{n+2}(c_1)=\max_{x\in[a,b]}|\phi_{n+2}(x)|>\max_{x\in[a,b]}|z(x)|$. Hence $\phi_{n+2}(x)$ is the desired solution.

EXAMPLES. (1) Let $L_{n+k}y=y^{(n+k)}=0$ $(k=0, 1)$, $a=-1$, $b=1$, and $\phi_i(x)=x^{i-1}$ $(i=1, 2, \cdots, n+1)$. In this case the solutions in question are polynomials of degree n with leading coefficient one. The function $\phi(x)$ of the theorem, as in well known, is the Tchebycheff polynomial $T_n(x)=(\frac{1}{2})^{n-1}\cos[n\cos^{-1}x]$.

(2) Let $n=2m+1$ and let

$$L_{n+k}y = y^{(n+k)} + \sum_{i=1}^{m+k/2} i^2 y^{(n+k-2)} + \sum_{j=2}^{m+k/2} \sum_{i<j} i^2 j^2 y^{(n+k-4)}$$
$$+ \sum_{m=3}^{m+k/2} \sum_{j<m} \sum_{i<j} i^2 j^2 m^2 y^{(n+k-6)} + \cdots$$
$$+ (1^2 \cdot 2^2 \cdot \ . \ . \ . \ . \cdot (m + k/2)^2) y' = 0,$$

where $(k=0, 2)$, $a=0$, $b=2\pi$ and $\phi_1(x)=1$, $\phi_2(x)=\sin x$, $\phi_3(x)=\cos x$, $\cdots$, $\phi_n(x)=\cos mx$, $\phi_{n+1}(x)=\sin (m+1)x$, $\phi_{n+2}(x)=\cos (m+1)x$. Thus we are considering the usual Fourier series partial sum of order $m+1$ with leading coefficient one. The function $\phi_{n+2}(x)=\cos (m+1)x$ is the function of the corollary.

Bibliography

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