# A FACTORIZATION OF STABLE HOMEOMORPHISMS OF $E^n$

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**Introduction.** In [2] it was shown that any stable homeomorphism of the $n$-sphere $S^n$ $(n>0)$ could be factored into the product of two topological dilations. This result was obtained by making repeated use of the invertibility properties of the sphere. A similar, though slightly weaker, proposition is true, and is proved in this paper, for the stable group on Euclidean $n$-space, $E^n$. However, since $E^n$ is not invertible, a procedure somewhat different from that employed in [2] must be followed here.

**Definitions and notation.** If $n$ is a positive integer, we denote by $E^n$ $n$-dimensional Euclidean space, by $G$ the group of stable homeomorphisms of $E^n$ onto itself, by $H$ the subgroup of $G$ consisting of all those homeomorphisms of $E^n$ onto itself each of which is supported on some $n$-cell, and by $\mathcal{C}$ we mean the collection of all images under the elements of $H$ of the unit $n$-cell in $E^n$. If $D\subset E^n$, we denote the interior of $D$ by $D^0$. If $f$, $g\in G$, then $gfg^{-1}$ is called a stable conjugate of $f$.

DEFINITION 1. If $p\in E^n$ and $\{D_i\}_{i=-\infty}^{\infty}=\mathfrak{D}\subset\mathcal{C}$, then the pair $(p, \mathfrak{D})$ is called a *dilation structure* on $E^n$, provided

(a) $D_i\subset D_{i+1}^0, \quad -\infty<i<\infty$,

(b) $\bigcap\mathfrak{D}=\{p\}$, and

(c) $\bigcup\mathfrak{D}=E^n$.

DEFINITION 2. An element $d$ of $G$ is called a *topological dilation* on $E^n$ if there exists a dilation structure $(p, \mathfrak{D})$ on $E^n$ such that $d(D_i)=D_{i+1}$, $-\infty<i<\infty$. In this case $(p, \mathfrak{D})$ is said to be a dilation structure for $d$, and $d$ is said to be carried by $(p, \mathfrak{D})$.

An example of a canonical topological dilation, $d$, is a homeomorphism of $E^n$ onto itself which expands radially outward from the origin. That is if $x\in E^n$, then $d(x)=rx$ where $r$ is a fixed real number greater than 1. Here $d$ is carried by the dilation structure $(p, \{d^i(C)\}_{i=-\infty}^{\infty})$, where $p$ is the origin and $C$ is the unit $n$-cell centered at the origin. Clearly any stable conjugate of $d$ is a topological dilation, and from Theorem 1 below we may conclude the converse: that any topological dilation is a stable conjugate of $d$.

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**Two lemmas and a theorem.**

LEMMA 1. *Suppose* $K_0$, $K_1 \in \mathcal{C}$ *with* $K_1 \subset K_0^0$ *and* $g \in G$ *with* $g(K_0) \subset K_1^0$. *Suppose also that* $\{D_i\}_{i=1}^{\infty} \subset \mathcal{C}$, $D_{i+1} \subset D_i^0$, $i \geqq 1$, $D_1 \subset K_1^0$, *and* $\bigcap \{D_i\}_{i=1}^{\infty}$ *is a single point. Then there exists* $h \in H$, *supported on* $K_1$ *such that* $(hg)^i(K_0) = D_i$, *for* $i \geqq 1$.

LEMMA 2. *Any dilation structure carries a topological dilation.*

PROOF. See the proofs of Lemma 2 and Lemma 3 in [2].

THEOREM 1. *Suppose* $d_1$ *and* $d_2$ *are topological dilations on* $E^n$. *Then there exists an* $f \in G$ *such that* $d_1 = fd_2f^{-1}$.

PROOF. The proof of this follows from Theorem 1 in [2] along with a couple of observations. If $h$ is a stable homeomorphism of $E^n$ onto itself then its natural extension to $S^n$ obtained by leaving fixed the point at infinity is stable on $S^n$; and conversely, if $h$ is stable on $S^n$ and leaves the point at infinity fixed, then $h$ restricted to $E^n$ is stable on $E^n$. Thus, if, in Theorem 1 in [2], one considers $g_1$ and $g_2$ to be the extensions to $S^n$ of $d_1$ and $d_2$ respectively, then the stable homeomorphism $r$ constructed there does leave the point at infinity fixed; and therefore we may take $f = r | E^n$.

THE FACTORIZATION THEOREM. *Suppose* $f \in G$ *and* $f$ *is not the identity. Then there exist topological dilations,* $d_1$ *and* $d_2$, *such that* $d_1 f = d_2$.

PROOF. Since $f$ is not the identity, there exists a point $p_1$ such that $p_1 \neq f(p_1) = p_2$. Select $C_0$, $D_1 \in \mathcal{C}$ with $p_1 \in C_0^0$, $C_0 \subset D_1^0 \supset f(C_0)$, and $f(C_0) \cap C_0 = \emptyset$. We may also select $\{D_i\}_{i=2}^{\infty} \subset \mathcal{C}$ with $\bigcup \{D_i\}_{i=1}^{\infty} = E^n$ and such that $D_i \subset D_{i+1}^0$ and $D_i \subset f(D_{i+1}^0)$ for $i \geqq 1$.

Now set $D_0 = f(C_0)$ and select $\{D_i\}_{i=-1}^{-\infty} \subset \mathcal{C}$ with $\bigcap \{D_i\}_{i=-1}^{-\infty} = \{p_2\}$ and $D_i \subset D_{i+1}^0$ for $i \leqq -1$. Since $(p_2, \{D_i\}_{i=-\infty}^{\infty})$ is a dilation structure, by Lemma 2 it carries a topological dilation $d$. That is $d(D_i) = D_{i+1}$ for $-\infty < i < \infty$.

Set $g = f^{-1}d^{-2}$ and consider $g(D_1)$: $d^{-2}(D_1) = D_{-1} \subset D_0^0 = f(C_0)$. Therefore $g(D_1) = f^{-1}d^{-2}(D_1) = f^{-1}(D_{-1}) \subset f^{-1}(D_0^0) = C_0$. Set $K_1 = g(D_1)$ and $K_0 = C_0$. Since $C_0 \subset D_1^0$, $g(C_0) \subset g(D_1^0) = K_1^0$. Hence $g(K_0) \subset K_1^0$. Now select $q \in K_1^0$ and $\{F_i\}_{i=1}^{\infty} \subset \mathcal{C}$ with $F_1 \subset K_1^0$, $\bigcap \{F_i\}_{i=1}^{\infty} = \{q\}$, and $F_{i+1} \subset F_i^0$ for $i \geqq 1$. By Lemma 1 there exists $h \in H$, supported on $K_1$, such that $(hg)^i(K_0) = F_i$ for $i \geqq 1$.

We now show that $r = d^2hfh^{-1}$ is a topological dilation carried by $(q, \{r^i(C_0)\}_{i=-\infty}^{\infty})$. Consider $r(C_0)$. Since $h$ is supported on $K_1$ and $K_1 \subset C_0^0$, $h^{-1}(C_0) = C_0$; also $f(C_0) = D_0$, $h(D_0) = D_0$, and $d^2(D_0) = D_2$. Therefore $r(C_0) = D_2$. Now we consider $r^2(C_0) = r(D_2)$. Since $h$ is supported on $K_1$, $h^{-1}(D_2) = D_2$; also $f(D_2) \supset D_1$. Thus $hf(D_2) = f(D_2)$.

Hence $r(D_2)=d^2f(D_2)\supset d^2(D_1)=D_3$. Continuing in this manner it is easy to see that $r^i(C_0)\supset D_i$ for $i\geqq 1$. From this and the fact that $C_0\subset r(C_0^0)$ we conclude that $r^i(C_0)\subset r^{i+1}(C_0)$ for $-\infty<i<\infty$ and $\bigcup\{r^i(C_0)\}_{i=-\infty}^{\infty}=E^n$. It only remains to show that $\{q\}=\bigcap\{r^i(C_0)\}_{i=0}^{-\infty}$.

Consider $r^{-1}(C_0)=hf^{-1}h^{-1}d^{-2}(C_0)$. Since $d^{-2}(C_0)\subset D_0$, $h^{-1}d^{-2}(C_0)=d^{-2}(C_0)$. Hence $r^{-1}(C_0)=hf^{-1}h^{-1}d^{-2}(C_0)=hf^{-1}d^{-2}(C_0)=hg(C_0)=F_1$. Now consider $r^{-2}(C_0)=r^{-1}(F_1)$. Again, since $d^{-2}(F_1)\subset D_0$, we have that $h^{-1}d^{-2}(F_1)=d^{-2}(F_1)$, and consequently $r^{-1}(F_1)=hf^{-1}h^{-1}d^{-2}(F_1)=hf^{-1}d^{-2}(F_1)=hg(F_1)=F_2$. Continuing in this manner we see that $r^{-i}(C_0)=F_i$ for $i\geqq 1$. Hence $\{q\}=\bigcap\{r^i(C_0)\}_{i=0}^{-\infty}$ and $r=d^{-2}hfh^{-1}$ is a topological dilation. Observing that $d_1=h^{-1}d^2h$ and $d_2=h^{-1}rh$ are topological dilations and that $d_1f=d_2$ completes the proof of the theorem.

COROLLARY. *Suppose $d$ is a topological dilation and $f$ is a stable homeomorphism. Then $f$ is the product of a stable conjugate of $d$ by a stable conjugate of $d^{-1}$.*

PROOF. If $f$ is the identity, then $f=dd^{-1}$. Suppose $f$ is not the identity. Then by Theorem 2 there are topological dilations $d_1$ and $d_2$ such that $d_1f=d_2$. By Theorem 1 there exist stable homeomorphism $r_1$ and $r_2$ such that $d_1=r_1dr_1^{-1}$ and $d_2=r_2dr_2^{-1}$. Hence $r_1dr_1^{-1}f=r_2dr_2^{-1}$, and therefore $f=(r_1d^{-1}r_1^{-1})(r_2dr_2^{-1})$.

## BIBLIOGRAPHY

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