CHROMATIC NUMBER OF CARTESIAN SUM OF TWO GRAPHS

KUNG-WEI YANG

In this note, we will consider the class of all finite undirected graphs with simple edges and no loops [1]. Let G, G_1, G_2 denote graphs. Let $o(G)$ = the number of vertices in G, $\beta(G)$ = the independence number of G, $\mathrm{K}(G)$ = the chromatic number of G, $G_1 \oplus G_2$ = the Cartesian sum of G_1 and G_2 [1]. We shall prove the following

THEOREM. $o(G_1)o(G_2)/\beta(G_1)\beta(G_2) \leqq \mathrm{K}(G_1 \oplus G_2) \leqq \mathrm{K}(G_1)\mathrm{K}(G_2)$.

Moreover, an example is given to show that the inequality $\mathrm{K}(G_1 \oplus G_2) < \mathrm{K}(G_1)\mathrm{K}(G_2)$ in fact occurs.

The proof is based on the following two lemmas.

LEMMA 1. $\beta(G_1 \oplus G_2) = \beta(G_1)\beta(G_2)$.

PROOF. If G is a graph, we denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G. We say that a subset $S \subset V(G)$ is independent if for any a, b in S, $(a, b) \notin E(G)$. If S is a set, we denote by $|S|$ the number of elements in S. Now, let $S_i \subset V(G_i)$ $(i = 1, 2)$ be independent sets such that $|S_i| = \beta(G_i)$. If $S = \{a_1 \oplus a_2 : a_1 \in S_1$ and $a_2 \in S_2\}$, then S is an independent set of $G_1 \oplus G_2$. Therefore, $\beta(G_1 \oplus G_2) \geqq |S| = \beta(G_1)\beta(G_2)$.

Suppose $T \subset V(G_1 \oplus G_2)$ is an independent set such that $|T| = \beta(G_1 \oplus G_2)$. For $a \in V(G_1)$, let $T(a) = \{b \in V(G_2) : a \oplus b \in T\}$. Then for each $a \in V(G_1)$, $T(a)$ is an independent set of G_2. Therefore, for each $a \in V(G_1)$, $|T(a)| \leqq \beta(G_2)$. Clearly, $|T| = \sum_{a \in V(G_1)} |T(a)|$. But $|T(a)| = 0$ except for those a in an independent set of G_1. Hence $\beta(G_1 \oplus G_2) = |T| \leqq \beta(G_1)\beta(G_2)$. This shows that $\beta(G_1 \oplus G_2) = \beta(G_1)\beta(G_2)$.

LEMMA 2. $\mathrm{K}(G_1)\mathrm{K}(G_2) \geqq \mathrm{K}(G_1 \oplus G_2)$.

PROOF. Let G, H be graphs. A homomorphism $f : G \to H$ is a map $f : V(G) \to V(H)$ such that if $(a, b) \in E(G)$, then $(f(a), f(b)) \in E(H)$. We say that a homomorphism $f : G \to H$ is surjective if $f : V(G) \to V(H)$ is surjective. If we denote by K_n the complete graph of n vertices, then the chromatic number of a graph G can be defined as the smallest integer n such that there exists a surjective homomorphism $f : G \to K_n$. Now let $m = \mathrm{K}(G_1)$ and $n = \mathrm{K}(G_2)$ and let $f_1 : G_1 \to K_m$ and $f_2 : G_2 \to K_n$ be surjective homomorphisms. If we define

Received by the editors February 14, 1967.

$$f_1 \oplus f_2 \colon G_1 \oplus G_2 \to K_m \oplus K_n$$

by

$$f_1 \oplus f_2(a_1 \oplus a_2) = f_1(a_1) \oplus f_2(a_2),$$

then $f_1 \oplus f_2$ is obviously a surjective homomorphism. Since $K_m \oplus K_n \cong K_{mn}$, we obtain $\mathrm{K}(G_1 \oplus G_2) \leqq \mathrm{K}(G_1) \cdot \mathrm{K}(G_2)$.

PROOF OF THE THEOREM. We only need to show that

$$(o(G_1)o(G_2)/\beta(G_1)\beta(G_2)) \leqq \mathrm{K}(G_1 \oplus G_2).$$

By [1, Theorem 14.1.1, p. 225], we have

$$(o(G_1 \oplus G_2)/\beta(G_1 \oplus G_2)) \leqq \mathrm{K}(G_1 \oplus G_2).$$

By Lemma 1, $\beta(G_1 \oplus G_2) = \beta(G_1)\beta(G_2)$. It is obvious that $o(G_1 \oplus G_2) = o(G_1)o(G_2)$. Therefore, $(o(G_1)o(G_2)/\beta(G_1)\beta(G_2)) \leqq \mathrm{K}(G_1 \oplus G_2)$.

Finally, we present an example to show that the inequality $\mathrm{K}(G_1 \oplus G_2) < \mathrm{K}(G_1)\mathrm{K}(G_2)$ actually occurs.

Let G_1 be the graph with

$$V(G_1) = \{1, 2, 3, 4, 5\}, \qquad E(G_1) = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}.$$

Let G_2 be the same graph with different labeling of the vertices, say,

$$V(G_2) = \{1', 2', 3', 4', 5'\},$$
$$E(G_2) = \{(1', 2'), (2', 3'), (3', 4'), (4', 5'), (5', 1')\}.$$

Clearly, $\mathrm{K}(G_1) = \mathrm{K}(G_2) = 3$. We now define a surjective homomorphism $f\colon G_1 \oplus G_2 \to K_8$, where $V(K_8) = \{k_1, k_2, \cdots, k_8\}$, by

$$f^{-1}[k_1] = \{1 \oplus 1', 1 \oplus 3', 3 \oplus 1', 3 \oplus 3'\},$$
$$f^{-1}[k_2] = \{1 \oplus 2', 1 \oplus 4', 3 \oplus 2', 3 \oplus 4'\},$$
$$f^{-1}[k_3] = \{2 \oplus 1', 2 \oplus 3', 4 \oplus 1', 4 \oplus 3'\},$$
$$f^{-1}[k_4] = \{2 \oplus 2', 2 \oplus 4', 4 \oplus 2', 4 \oplus 4'\},$$
$$f^{-1}[k_5] = \{3 \oplus 5', 5 \oplus 3', 5 \oplus 5'\},$$
$$f^{-1}[k_6] = \{2 \oplus 5', 5 \oplus 2'\},$$
$$f^{-1}[k_7] = \{1 \oplus 5', 4 \oplus 5'\},$$
$$f^{-1}[k_8] = \{5 \oplus 1', 5 \oplus 4'\}.$$

The example shows that $\mathrm{K}(G_1 \oplus G_2) \leqq 8 < 9 = \mathrm{K}(G_1)\mathrm{K}(G_2)$.

BIBLIOGRAPHY

1. O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ., Vol. 38, Amer. Math. Soc., Providence, R. I., 1962.

WESTERN MICHIGAN UNIVERSITY