A BOL LOOP ISOMORPHIC TO ALL LOOP ISOTOPES

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1. **Introduction.** A loop $(G, \cdot)$ is a *Bol loop* if and only if

$$(1) \qquad (xy\cdot z)y = x(yz\cdot y)$$

for all $x, y, z \in G$. Bol loops had their origin in the work of Bol [1] and their basic algebraic properties are discussed by Robinson [2], [3]. In [3] the following question is raised: If a Bol loop $(G, \cdot)$ is isomorphic to each of its loop isotopes, is $(G, \cdot)$ necessarily Moufang? The purpose of this note is to answer the preceding question by actually constructing a Bol loop $(G, \cdot)$ which is isomorphic to all loop isotopes but which fails to be Moufang.

2. **Construction.** Let R be an alternative division ring which is not associative and which is not of characteristic 2. Let G be the Cartesian product $G = R\times R\times R\times R\times R$. If $x = (a_1, b_1, c_1, d_1, e_1)$ and $y = (a_2, b_2, c_2, d_2, e_2)$ are elements of G, define $x\cdot y$ and $x - y$ by

$$(2) \quad x\cdot y = (a_1+a_2,\ b_1+b_2,\ c_1+c_2+a_1b_2,\ d_1+d_2+b_1b_2,\ e_1+e_2+a_1d_2+c_1b_2)$$

and

$$(3) \qquad x - y = (a_1 - a_2,\ b_1 - b_2,\ c_1 - c_2,\ d_1 - d_2,\ e_1 - e_2).$$

We now proceed to show that the resulting binary system $(G, \cdot)$ satisfies the conditions stated in the introduction.

RESULT 1. $(G, \cdot)$ is a Bol loop.

PROOF. Clearly $(G, \cdot)$ is a loop with identity $(0, 0, 0, 0, 0)$. Now let $x = (a_1, b_1, c_1, d_1, e_1)$, $y = (a_2, b_2, c_2, d_2, e_2)$ and $z = (a_3, b_3, c_3, d_3, e_3)$ be elements of G. Using (2) and (3), we see that

$$(xy\cdot z)y - x(yz\cdot y) = (0, 0, 0, 0, a_1b_2\cdot b_3 + a_1b_3\cdot b_2 - a_1\cdot b_2b_3 - a_1\cdot b_3b_2).$$

Since R is right alternative, we have

$$a_1(b_2 + b_3)^2 = [a_1(b_2 + b_3)](b_2 + b_3).$$

Upon expanding, we see that $a_1b_2\cdot b_3 + a_1b_3\cdot b_2 - a_1\cdot b_2b_3 - a_1\cdot b_3b_2 = 0$. Thus, $(xy\cdot z)y = x(yz\cdot y)$ and $(G, \cdot)$ is a Bol loop.

RESULT 2. $(G, \cdot)$ is not Moufang.

PROOF. Since R is not associative, there exist elements $a, b, c \in R$ such that $ab\cdot c \neq a\cdot bc$. Now let $u = (a, c, 0, 0, 0)$ and $v = (0, b, 0, 0, 0)$.

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Then, using (2) and (3), we obtain

$$uv\cdot u - u\cdot vu = (0, 0, 0, 0, ab\cdot c - a\cdot bc).$$

Thus, $uv\cdot u \neq u\cdot vu$. Hence, $(G, \cdot)$ is not di-associative and, consequently, is not Moufang.

RESULT 3. If the loop $(H, *)$ is isotopic to $(G, \cdot)$, then there is an element g in the left nucleus of $(G, \cdot)$ so that $(H, *)$ is isomorphic to $(G, \circ)$ where

$$x \circ y = xR(g\cdot g)\cdot yL(g\cdot g)^{-1}.$$

PROOF. There is an element $f \in G$ so that $(H, *)$ is isomorphic to $(G, \otimes)$ where $x \otimes y = xR(f)\cdot yL(f)^{-1}$. (See [3, Lemma 3.4].) Let $f = (a, b, c, d, e)$. Then, if $x = (a_1, b_1, c_1, d_1, e_1)$ and $y = (a_2, b_2, c_2, d_2, e_2)$, we get

$$(x \otimes y) - (x\cdot y) = (0, 0, 0, 0, [a_1, b, b_2])$$

where $[a_1, b, b_2]$ is the ring associator $[a_1, b, b_2] = a_1b\cdot b_2 - a_1\cdot bb_2$. Since R is an alternative division ring which is not of characteristic 2, we can choose an element $\bar{b} \in G$ so that $2\bar{b} = b$. Now let $g = (0, \bar{b}, 0, 0, 0)$. It is easy to see that g is in the left nucleus of $(G, \cdot)$ and, furthermore, that $x \otimes y = xR(g\cdot g)\cdot yL(g\cdot g)^{-1}$.

RESULT 4. Every loop isotopic to $(G, \cdot)$ is isomorphic to $(G, \cdot)$.

PROOF. In view of Result 3, it suffices to show that $(G, \cdot)$ is isomorphic to $(G, \circ)$ where $x \circ y = xR(g\cdot g)\cdot yL(g\cdot g)^{-1}$ for g in the left nucleus of $(G, \cdot)$. Since $(G, \cdot)$ is a Bol loop, $(R(g)^{-1}, L(g)R(g), R(g))$ is an autotopism of $(G, \cdot)$ (see [3, Theorem 2.3]) and, since g is in the left nucleus of $(G, \cdot)$, we see that $(L(g), I, L(g))$ is also an autotopism of $(G, \cdot)$ where I is the identity mapping on G. Thus,

$$\begin{aligned}(L(g), I, L(g))&(R(g)^{-1}, L(g)R(g), R(g)) \\ &= (L(g)R(g)^{-1}, L(g)R(g)^{-1}R(g)^2, L(g)R(g)^{-1}R(g)^2)\end{aligned}$$

is an autotopism of $(G, \cdot)$. From (1) it follows that $R(g)^2 = R(g\cdot g)$. Thus, $g\cdot g$ is a companion of a pseudo-automorphism of $(G, \cdot)$. Hence, by [3, Theorem 3.3], we conclude that $(G, \cdot)$ and $(G, \circ)$ are isomorphic.

REFERENCES

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