THE OPEN CASE FOR SIMPLE ALTERNATIVE RINGS

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1. **Introduction.** Let R be a simple alternative ring, not associative, and not a Cayley-Dickson algebra over its center. It was proved by Kleinfeld ([**2**] and [**3**]) that R is necessarily nil and of char 3. It has remained an open question whether such a ring R can exist.

We show here, using a result given by Shirshov (in [**6**]; see also [**5**]), that such a ring cannot exist (Theorem C).

2. **Shirshov's result.** We now state a convenient version of one of Shirshov's theorems. Let X be any set (of 'indeterminates'). Let $N_X(F)$ and $S_X(F)$ be, respectively, the free nonassociative and free associative algebras over the field F and on the free generating set X. There is an obvious F-homomorphism α from $N_X(F)$ onto $S_X(F)$: in the standard representation α deletes all parentheses. Say $p \in N_X(F)$ is *admissible* provided $p\alpha \neq 0$.

Now suppose R is any (nonassociative) algebra over F. Say R *satisfies the admissible polynomial identity* (p.i.) p, provided $p \in N_X(F)$ is admissible, and for every F-homomorphism β of $N_X(F)$ into R, $p\beta = 0$.

We can now state

THEOREM A (SHIRSHOV).[1] *Let R be an alternative algebra over a field F of char $\neq 2$. Suppose R satisfies an admissible p.i. If R is algebraic over F, then R is locally finite over F.*

COROLLARY 1. *Under the hypotheses of Theorem A, if R is nil, then R is locally nilpotent.*

PROOF. Let T be any finitely generated subalgebra of R. Then by Theorem A, T is finite dimensional over F. But it is a classical result (e.g. see [**4**, p. 30]) that a nil finite-dimensional alternative algebra is nilpotent. So T is nilpotent, and R is locally nilpotent.

In view of Corollary 1, we now turn our attention to locally nilpotent rings.

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[1] The relevant result is Theorem 5 of [6]. The statement of this theorem contains some errors, and the proof outlined is rather obscure. A set of notes giving a full exposition of the relevant results has been prepared by the author of the present paper, and is available from him.

3. **Locally nilpotent rings.** Let R be either a ring or an algebra over a field F. R need not be associative. We say R is *locally nilpotent* provided every finitely generated subring (respectively subalgebra) is nilpotent.

LEMMA 2. *Let R be a locally nilpotent ring or algebra. Given $a \in R$, let A be the ideal of R generated by $aR+Ra$. If $a \in A$, then $a=0$.*

PROOF. We first introduce some notation. Let $E(R)$ be the endomorphism ring (or algebra) of R. Given $s \in R$, we define $L(s)$ and $R(s) \in E(R)$ by $x \cdot R(s) = xs$; $x \cdot L(s) = sx$ (all $x \in R$). We write $U(s)$ to denote ambiguously either $R(s)$ or $L(s)$.

If $(s, n) = (s_1, s_2, \cdots, s_n)$ is an n-tuple of elements of R ($n \geqq 1$), we write $U[s, n]$ for any $V \in E(R)$ of the form $V = U(s_1) \cdot U(s_2) \cdot \cdots \cdot U(s_n)$. Thus V is the product of n right or left multiplications.

We define addition on the set of tuples by setting $(s, n)+(s', m) = (s_1, s_2, \cdots, s_n, s_1', s_2', \cdots, s_m') = (s+s', n+m)$, say. Then it is immediate that any product $U[s, n] \cdot U[s', m]$ is some $U[s+s', n+m]$.

Now in any ring or algebra R the ideal generated by $aR+Ra$ (for given $a \in R$) is easily seen to be the set of all finite sums of the form

$$\sum_i a \cdot U[s(i), n(i)], \qquad n(i) \geqq 1; \ (s(i), n(i)) = (s(i)_1, \cdots, s(i)_{n(i)}).$$

In particular, if $a \in A$, we can write

$$\text{(i)} \qquad a = \sum_{i=1}^{r} a \cdot U[s(i), n(i)], \qquad n(i) \geqq 1.$$

Set

$$T = \{a\} \cup \{s(i)_{j(i)} : 1 \leqq j(i) \leqq n(i); 1 \leqq i \leqq r\}.$$

We now claim that for any given $m \geqq 1$ we can write a in the form

$$\text{(ii)} \qquad a = \sum_{i=1}^{t} a \cdot U[s(i), n(i)], \qquad n(i) \geqq m,$$

where $t = t_m$, $s(i) = s_m(i)$, $n(i) = n_m(i)$ are all functions of m, and every $s(i)_{j(i)} \in T$.

For $m=1$ such an expression is given in (i) (by definition of T). We prove the general assertion by induction on m. Suppose we have an expression (ii) for a and given $m \geqq 1$. Then, substituting the expression (ii) for a in the right side of the equality (ii), we obtain

$$
\begin{aligned}
a &= \sum_{i=1}^{t} \left\{ \sum_{j=1}^{t} a \cdot U[s(j), n(j)] \right\} \cdot U[s(i), n(i)] \\
&= \sum_{i=1}^{t} \sum_{j=1}^{t} a \cdot \{ U[s(j), n(j)] \cdot U[s(i), n(i)] \} \\
&= \sum_{i,j=1}^{t} a \cdot U[s(j) + s(i), n(j) + n(i)].
\end{aligned}
$$

Since $n(j)+n(i) \geqq m+m \geqq m+1$, this expression for a is of the form (ii) with $m+1$ in place of m. This completes the induction.

Now let S be the subring or subalgebra of R generated by T. Since T is a finite set, S is nilpotent, say of index m. If $a \in A$, then by (ii) we can write a as a sum of terms $a \cdot U[s, n]$; $n \geqq m$. But $a \cdot U[s, n]$ is a product of $\geqq n+1 \geqq m$ elements of T, and so lies in $S^m = (0)$. Thus a is a sum of zero terms, and $a=0$.

As an easy corollary we have

THEOREM B. *Let R be a simple (nonassociative) ring. Then R is not locally nilpotent.*

PROOF. Suppose R is simple and locally nilpotent. Let $0 \neq a \in R$ be arbitrary, and A as in Lemma 2. Then $a \notin A$, so $A \neq R$, whence $A = (0)$. Thus a is a total zero-divisor. So R consists entirely of zero-divisors, and thus is a zero ring. Such rings are not regarded as being simple.

NOTE. This proof was inspired by that given in [**1**, p. 31] for associative rings.

4. Simple alternative rings. We start by recalling some known results. They may all be found in [**3**]. Let R be an alternative ring, $N(R)$ its nucleus, $Z(R)$ its center. For $x, y \in R$, write (x, y) for $xy - yx$. Then we have

LEMMA 3 (KLEINFELD). *For all $x, y \in R$, $(x, y)^4 \in N(R)$. If R is prime, $N(R) = Z(R)$. If R is simple, but neither associative nor a Cayley-Dickson algebra over $Z(R)$, then $3R = (0)$ and R is nil.*

We now have

THEOREM C. *If R is a simple alternative ring, then R is either associative or a Cayley-Dickson algebra over $Z(R)$.*

PROOF. Suppose R is neither. Then by Lemma 3 we may regard R as a nil algebra over the field J_3 of char $\neq 2$. Since simple rings are prime, it also follows from Lemma 3 that R satisfies the identity $p=p(x, y, z)=((x, y)^2(x, y)^2, z)\in N_X(J_3)$. Since p is admissible, R is locally nilpotent by Corollary 1. By Theorem B such an R cannot exist.

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